1. The value of $1 - \frac{1}{1-\frac{1}{1-\frac{1}{11}}}$ is:

(**A**∗) 11   (B) 0   (C) -3 (D) -9 (E) -10

Solution

We simplify one layer at a time!

$$1 - \frac{1}{1 - \frac{1}{1-\frac{1}{11}}} = 1 - \frac{1}{1 - \frac{1}{\frac{11}{11}-\frac{1}{11}}} = 1 - \frac{1}{1 - \frac{1}{\frac{10}{11}}} = 1 - \frac{1}{1 - \frac{11}{10}} = 1 - \frac{1}{\frac{10}{10} - \frac{11}{10}} = 1 - \frac{1}{-\frac{1}{10}} = 1 + 10 = 11$$

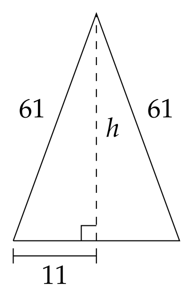
Answer: A

2. Two sides of an isosceles triangle are 22 and 61. The area of the triangle is:

(A) 620   (B) 638   (C) 649   (**D**∗) 660   (E) 671

Solution

The two equal sides of the triangle must have length 61 (since 22 is too short). So we can draw the triangle as follows:



By the Pythagorean theorem, we have

$$11^2 + h^2 = 61^2,$$

which gives

$$h^2 = 61^2 - 11^2 = 3721 - 121 = 3600,$$

so $h = 60$. Thus, the area of the triangle is given by

$$A = \frac{bh}{2} = \frac{22(60)}{2} = 660.$$

Answer: D

3. Madeleine checks the odometer on her car, which reads 78, 987 kilometers. She notices the number is a palindrome: it reads the same backward and forward. If she is driving at a speed of 75 kilometers per hour then the amount of time before the odometer shows the next palindrome number will be:

(A) under 1 hour   (**B**∗) between 1 and 2 hours   (C) between 2 and 3 hours

(D) between 3 and 4 hours   (E) over 4 hours

Solution

The next palindrome after 78987 is 79097. So Madeleine will need to travel

$$79097 - 78987 = 110$$

kilometers before the odometer shows the next palindrome. Since she is driving at 75 kilometers per hour, this will take between 1 and 2 hours. (More precisely, it will take $\frac{110}{75}$ hours, or 88 minutes.)

Answer: B

4. If the following equations are true

$$A + B = 1$$
$$B + C = 2$$
$$C + D = 3$$

then $A + D$ equals:

(A) -1 (B) 1 (**C**$*$) 2 (D) 3 (E) -3

Solution

From the first and third equations we obtain $A + B + C + D = 4$. Since $B + C = 2$, we must have $A + D = 2$.

Answer: C

5. In a certain city, a taxi charges 0.20$ per 1/5 km traveled when moving faster than x km/h. It charges 0.15$ per minute when moving slower than x km/h. At x km/h, both methods of charging produce the same cost to the rider. The value of x is:

(**A**$*$) 9 (B) 10 (C) 12 (D) 15 (E) 18

Solution

When the taxi is moving at exactly x km/h, the cost per hour under the first method is given by

$$\frac{\$0.20}{\frac{1}{5}\text{ km}} \cdot \frac{x\text{ km}}{\text{h}} = \$x/\text{h}.$$

Meanwhile, the cost per hour under the second method is given by

$$\frac{\$0.15}{1\text{ min}} \cdot \frac{60\text{ min}}{1\text{ h}} = \$9/\text{h}.$$

So we see that $x = 9$.

Answer: A

6. Students in an art class are tiling the floor of their classroom with square tiles (1 inch by 1 inch). On the first day, they place one tile in the centre of the floor. On the second day, they surround that first tile with eight more, to make 3-by-3 square. On the third day, this becomes a 5-by-5 square, etc. The number of tiles they use on the 18th day is:

(A) 124 (B) 132 (**C**$*$) 136 (D) 140 (E) 144

Solution

We make a table that shows the *total* number of tiles on the floor at the end of each day, along with the number of *new* tiles used on each day, for the first five days.

3	3	3	3	3
3	2	2	2	3
3	2	1	2	3
3	2	2	2	3
3	3	3	3	3

Day	1	2	3	4	5
Total	1	9	25	49	81
New	1	8	16	24	32

We notice that when $n \geq 2$, the number of new tiles used on day n is $8n - 8$. Therefore, the number of tiles used on the 18th day is

$$8(18) - 8 = 144 - 8 = 136.$$

For a more careful justification that this pattern holds, note that on day n, the students form a $(2n - 1) \times (2n - 1)$ square, so the total number of tiles on the floor at the end of day n is $(2n - 1)^2$. It follows that when $n \geq 2$, the number of new tiles used on day n is

$$\underbrace{(2n-1)^2}_{\text{day } n \text{ total}} - \underbrace{(2n-3)^2}_{\text{day } n-1 \text{ total}} = 4n^2 - 4n + 1 - (4n^2 - 12n + 9) = 8n - 8.$$

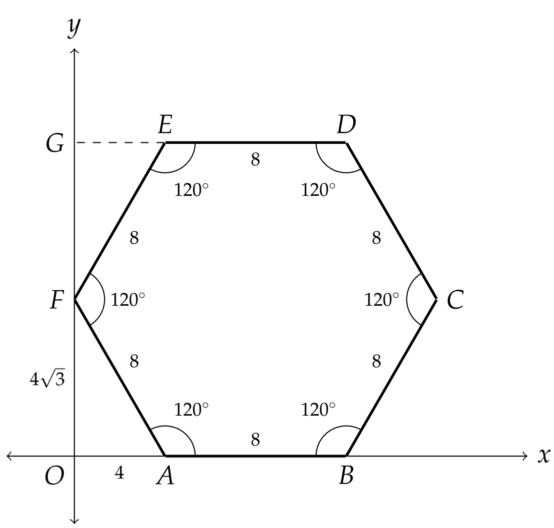
Answer: C

7. The point $A(4, 0)$ is a vertex of regular hexagon $ABCDEF$, whose side is 8 and whose interior lies completely within Quadrant 1. If vertex D has coordinates (x, y) then (x, y) is:

(A) $(8, 8\sqrt{3})$ (B) $(4, 4\sqrt{3})$ (C) $(16, 4\sqrt{3})$ (D) $(12, 4\sqrt{3})$ (**E**$*$) $(12, 8\sqrt{3})$

Solution

First we notice that the hexagon must appear as in the following diagram, with B on the x-axis, and F on the y-axis. (We label the vertices counterclockwise starting from A, but notice that D would appear in the same location if we went in the opposite order.)

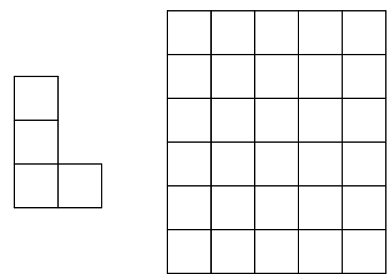


For suppose that F lies on the y-axis, as in this diagram. Since OA has length 4 and AF has length 8, it follows from the pythagorean theorem that OF has length $\sqrt{8^2 - 4^2} = \sqrt{48} = 4\sqrt{3}$. But then $\triangle OAF$ is a special triangle (as a right triangle with side lengths $4 \cdot 1$, $4 \cdot \sqrt{3}$, and $4 \cdot 2$). So we have $\angle OAF = 60^\circ$. Since $\angle FAB = 120^\circ$, we see that B must lie on the x-axis. It follows that F *must* lie on the y-axis (and in turn, B must lie on the x-axis) – otherwise, if F were in the first quadrant, then B would be pushed into the fourth quadrant!

Let G be the point where the extension of DE intersects the y-axis, as in the diagram. We see that $\triangle GFE$ is congruent to $\triangle OAF$. It follows that G has coordinates $(0, 8\sqrt{3})$, and that GD is horizontal and has length 12, so D has coordinates $(12, 8\sqrt{3})$.

Answer: E

8. The cutout shown is used to cover exactly four of the squares on the 5×6 checkerboard shown on the right. If rotations of the cutout are allowed, but not reflections, then the number of different choices for the four squares covered is:

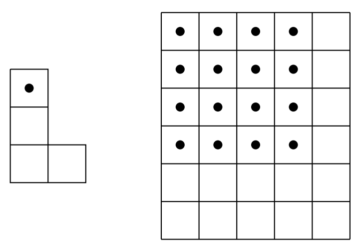


(A) 56 (B) 58 (C) 60

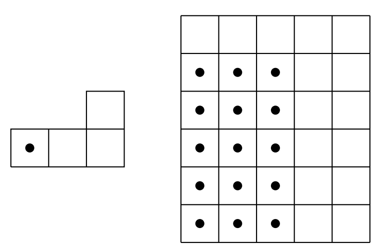
(**D**$*$) 62 (E) 64

Solution

We see that there are $4 \cdot 4 = 16$ ways to place the cutout without rotating:



If we rotate the cutout 90° counterclockwise, then there are $3 \cdot 5 = 15$ ways to place the cutout:



Rotating twice more, we get the same two pictures above but rotated 180°. So the total number of different choices is

$$2(16) + 2(15) = 62.$$

Answer: D

9. John throws a fair 6-sided die. If it lands showing a number 4 or more, he wins. If not, he throws again and wins if it lands showing a number 5 or more; if not, he throws again and wins only if it lands

showing a 6. The probability that John wins is:

(A) $\frac{2}{3}$ (B) $\frac{3}{4}$ (C) $\frac{5}{6}$ (D) $\frac{1}{12}$ (**E**∗) $\frac{13}{18}$

Solution

The probability that John wins is 1 minus the probability that John *loses* on all three rolls, which is

$$1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{5}{6} = 1 - \frac{5}{18} = \frac{13}{18}.$$

Answer: E

10. The smallest value of n for which the product

$$10^{\frac{1}{7}} \times 10^{\frac{2}{7}} \times 10^{\frac{3}{7}} \times 10^{\frac{4}{7}} \times \cdots \times 10^{\frac{n}{7}}$$

exceeds 2020 is:

(A) 6 (**B**∗) 7 (C) 8 (D) 10 (E) 12

Solution

Clearly the product is an increasing function of n. Since $n = 6$ gives the product 1000 and $n = 7$ gives the product $10,000$, the answer is $n = 7$.

Note: We have

$$\begin{aligned} 10^{\frac{1}{7}} \times 10^{\frac{2}{7}} \times 10^{\frac{3}{7}} \times 10^{\frac{4}{7}} \times \cdots \times 10^{\frac{n}{7}} &= 10^{\frac{1}{7}+\frac{2}{7}+\frac{3}{7}+\cdots+\frac{n}{7}} \\ &= 10^{\frac{1}{7}(1+2+3+\cdots+n)} \\ &= 10^{\frac{1}{7}\cdot\frac{n(n+1)}{2}}, \end{aligned}$$

which facilitates a quick calculation of the product for any given number n.

Answer: B

11. A number has the "increasing digits" property if its digits increase from left to right. For example, the numbers 189 and 378 have this property but the numbers 814 and 533 do not. The number of integers between 100 and 999 with the "increasing digits" property is:

(A) 48 (B) 60 (C) 72 (**D**∗) 84 (E) 96

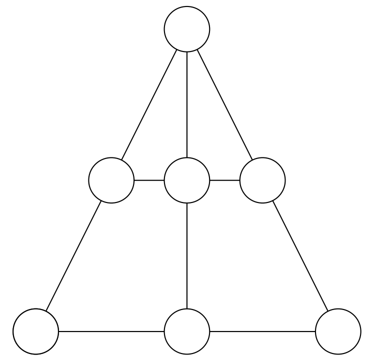
Solution

Each integer between 100 and 999 satisfying the "increasing digits" property contains three unique digits from $\{1, 2, \ldots, 9\}$, and each set of three unique digits from $\{1, 2, \ldots, 9\}$ determines exactly one such number. So the number of integers between 100 and 999 satisfying the "increasing digits" property is exactly the number of ways to choose three unique digits from $\{1, 2, \ldots, 9\}$, which is

$$\binom{9}{3} = \frac{9!}{6! \cdot 3!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84.$$

Answer: D

12. The accompanying diagram contains several sets of circles that "line up" (3 circles to a line). There are 5 such "lines". The integers from 1 through 7 are to be inserted, one number to a circle, so that the sum of the three numbers in each line is the same (this can be done in many ways). The number that can **not** be placed in the lower left circle is:



(A) 1 (B) 2 (C) 3

(**D**∗) 4 (E) 5

Solution

Suppose that the numbers 1 through 7 have been placed in the circles so that the sum of the three numbers on each line is the same, and call this sum y. Let x be the number placed in the circle at the top. Note that the circle at the top is contained in three lines, while every other circle is contained in only two lines. Thus, summing the numbers in all five lines in two ways, we have

$$5y = 2(1+2+3+4+5+6+7) + x \quad \Rightarrow \quad 5y = 56 + x.$$

So $56 + x$ must be a multiple of 5, hence we have $x = 4$. Since we started with an arbitrary placement of the numbers satisfying the required property, we conclude that 4 must *always* be placed at the top in any such placement, hence it can *not* be placed in the lower left circle.

Answer: D

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