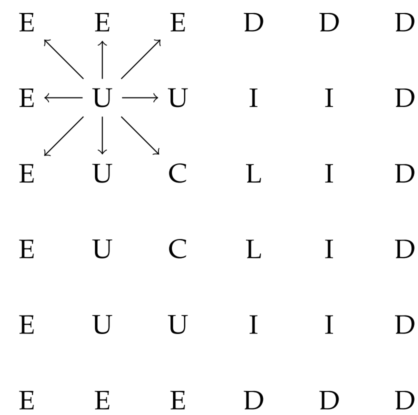
BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2023

Senior Final, Part B Problems & Solutions

1. The word EUCLID can be spelled by tracing paths through the given array of letters.

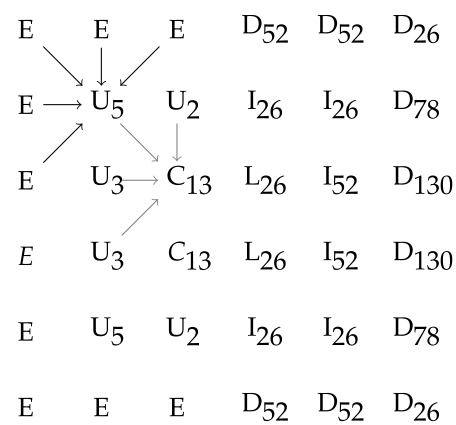
As shown in the diagram, steps to adjacent letters horizontally, vertically, or diagonally are allowed.



Determine the number of different paths which spell the word EUCLID.

Solution

The subscript n of a letter X_n indicates n choices for the previous letter to lead to X. For example, the upper left U_5 shows that five E's lead to that U. The upper C_{13} gets its subscript 13 from adding the subscripts of U that leads to this C, thus $2+5+3+3=13$. In short, for the number of paths which spell the word EUCLID, one finds the sum of all subscripts of D, namely, $(52+52+26+78+130)\times 2 = 676$ using a horizontal symmetry.



Answer: 676

2. What number leaves the same non-zero remainder when divided into 1108, 1453, 1844, and 2258?

Solution

We know $1108 = an + r$, $1453 = bn + r$, $1844 = cn + r$, and $2258 = dn + r$. Therefore, if we subtract any two of our numbers, we should get a multiple of our number n.The differences of our pairs of numbers are 345, 414, 391, and 1150. The greatest common divisor of those numbers is 23.

Answer: 23

3. Raul rolls two normal 6-sided dice, each die showing $\{1, 2, 3, 4, 5, 6\}$, and adds the numbers showing on them to get a sum of at least 2 and at most 12. He then raises the sum to the power of 4. The result is a number whose units digit is a 6.

   (a) What is the probability that the original sum was 8?

   (b) What is the probability that at least one of the dice that he rolled showed a 4?

Solution

(a) The sum is one of the numbers $2, 3, \ldots, 12$. Out of these sums, those whose fourth power results in a number ending in 6 are 2, 4, 6, 8 and 12. The number of ways that each of these occurs is 1, 3, 5, 5 and 1 so the total number of ways is 15. Out of 15 possibilities, 5 of them result in a sum equal to 8 namely, $(2, 6), (3, 5), (4, 4), (5, 3)$ and $(6, 2)$, so the probability is $\frac{5}{15} = \frac{1}{3}$.

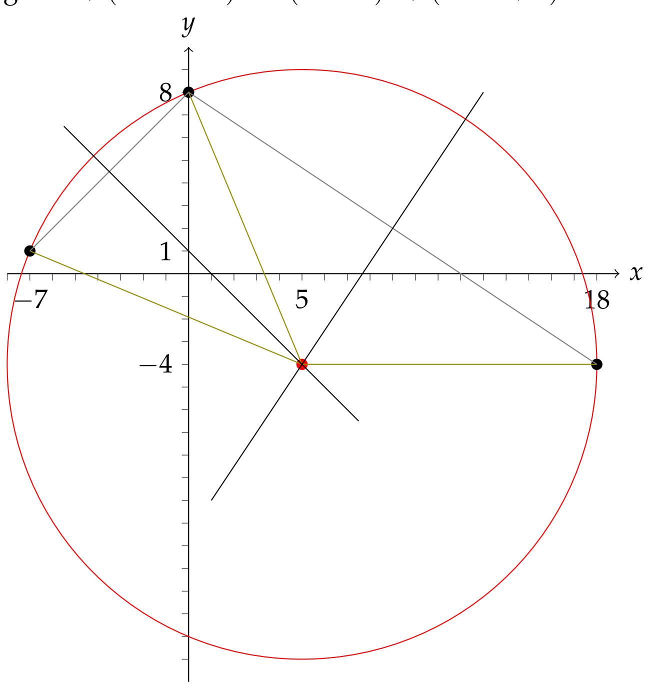
(b) The total number of ways that the fourth power of sum has an ending digit of 6 is still 15. Out of 15 possible rolls, three of them have at least a 4, namely $(2, 4), (4, 2)$ and $(4, 4)$, so the probability is $\frac{3}{15} = \frac{1}{5}$.

Answer: (a) $\frac{1}{3}$ (b) $\frac{1}{5}$

4. The three points $(-7, 1)$, $(0, 8)$, and $(18, -4)$ in the Cartesian plane lie on the boundary of a circle. Determine the location of the centre of the circle.

Solution

Solution1: If (h, k) is the centre of the circle, it must be the same distance r = radius from all the points on the perimeter. Plugging the first two points into the circle (or distance) formula, we get $(7 - h)^2 + (1 - k)^2 = h^2 + (8 - k)^2$. We solve to get $h + k = 1$ or $k = 1 - h$ so the centre can be expressed as $(h, 1 - h)$. Now we plug those coordinates in for the second two points on the circle, and get $h^2 + (1 - h - 8)^2 = (h - 18)^2 + (1 - h + 4)^2$. We can now solve to get $h = 5$, so $k = 1 - 5 = -4$.



Solution2: One can also use the intersection of two perpendicular bisectors of the line segment joining points $(-7, 1)$, $(0, 8)$ and the line segment joining $(0, 8)$, and $(18, -4)$ because the resulting point of intersection is equidistant to all three given points. The perpendicular bisector to the first line segment is the line $y - 9/2 = -(x + 7/2)$. That of the second line segment is $y - 2 = \frac{3}{2}(x - 9)$. Solving the system of two linear equations in x and y yields $x = 5$ and $y = -4$.

Answer: (5,-4)

5. If for any real number x, we have that $xf(x) + f(1 - x) = x^2 + 2$. Find each of the following.

(a) $f(0)$
(b) $f(5)$
(c) $f(19)$
(d) A formula for $f(x)$
(e) Show that f is unique.

Solution

(a) To find $f(0)$, substitute $x = 0$ in the given equation to get $f(1) = 2$. Next, substitute $x = 1$ to obtain $f(1) + f(0) = 3$. Since $f(1) = 2$, one gets $f(0) = 1$.

(b) For $f(5)$, substitute $x = 5$ to get the first equation, and $x = -4$ for the second.

$$5f(5) + f(-4) = 27, \quad -4f(-4) + f(5) = 18.$$

Or $21f(5) = 14 \times 9$, so $f(5) = 6$.

(c) Similarly, for $f(19)$, substitute $x = 19$, and $x = -18$ to get a system of two linear equations in $f(19)$ and $f(-18)$; thus, $f(19) = 20$.

(d) One can guess that $f(x) = x + 1$ from the pattern just established, then substitute into the original equation to check.

$$LHS = x(x + 1) + 2 - x = x^2 + x + 2 - x = x^2 + 2 = RHS$$

(e) Solution 1: In general, a formula for $f(x)$ can be obtained by substituting $x = n + 1$ and $x = -n$ to obtain a system of two linear equations in $f(n + 1)$ and $f(-n)$.

$$(n + 1)f(n + 1) + f(-n) = (n + 1)^2 + 2, \quad -nf(-n) + f(n + 1) = (-n)^2 + 2.$$

Multiply the first equation by n, then add the resulting equation to the second equation, one obtains

$$n(n + 1)f(n + 1) + f(n + 1) = n((n + 1)^2 + 2) + n^2 + 2, \quad \text{equivalently}$$
$$(n^2 + n + 1)f(n + 1) = (n^2 + n + 1)(n + 2), \quad \text{so} \quad f(n + 1) = n + 2,$$

or in terms of x, $f(x) = x + 1$. This shows that it is unique.

Solution 2: Let f and g both satisfy the given functional equation, namely,

$$xf(x) + f(1 - x) - (x^2 + 2) = xg(x) + g(1 - x) - (x^2 + 2).$$

Let $h(x) = f(x) - g(x)$. We get

$$xh(x) = -h(1 - x), \quad \text{also} \quad (1 - x)h(1 - x) = -h(x)$$

Therefore,

$$(1 - x)h(1 - x) = \frac{h(1 - x)}{x}$$

Since this equation is true for all x, the function h must equal to 0, or equivalently, $f = g$. Thus, f is unique.

Answer: (a) $f(0) = 1$ **(b)** $f(5) = 6$ **(c)** $f(19) = 20$ **(d)** $f(x) = x + 1$