BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2018

Senior Preliminary Problems & Solutions

1. Jar A contains flour and sugar in the ratio 5 : 1. Jar B, which is three times larger than Jar A, contains flour and sugar in the ratio 8 : 1. When the contents of these jars are combined, the resulting mixture contains flour and sugar in the ratio $x : 1$. The value of x is:

   (A) 6 (B) $\frac{13}{2}$ (C) $\frac{47}{7}$ (D) 7 (E) $\frac{29}{4}$

   Solution

   Jar A has 5 units flour, 1 unit sugar. Jar B is 3 times larger (so 18 units total) with ration 8 : 1, so 16 units flour, 2 units sugar. The combined ratio is: $5 + 16 : 1 + 3$ or $21 : 3 = 7 : 1$.

   Answer: D

2. Consider the statement: "In a room with exactly N people, you are guaranteed to be able to find at least three different people who are born in the same month." The smallest possible value of N that makes this statement true is:

   (A) 24 (B) 25 (C) 35 (D) 36 (E) 37

   Solution

   If you have 24 people, you can (barely) avoid having three with same birth month by having two people born in each separate month. The 25th person will have to "triple up".

   Answer: B

3. We say a positive integer is "happy" if it is less than 100 and is divisible by either 3 or 7, or both. For example, $3, 70$ and 84 are all happy. The number of happy numbers is:

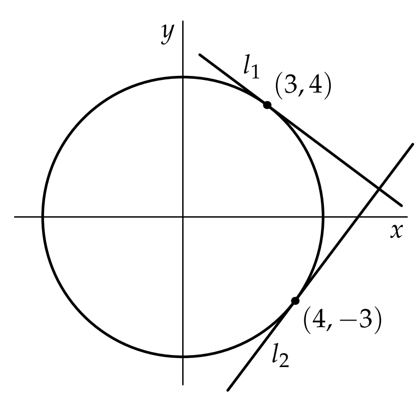
   (A) 39 (B) 40 (C) 43 (D) 45 (E) 47

   Solution

   The number of values less than 100 and divisible by 3 equals 33. The number of values less than 100 and divisible by 7 equals 14. The number of values divisible by both is 4. We have $33 + 14 - 4 = 43$.

   Answer: C

4. In the diagram shown, l_1 and l_2 are lines that are tangent to the circle $x^2 + y^2 = 25$ at the points $(3, 4)$ and $(4, -3)$. Let (p, q) be the coordinates of the point of intersection of l_1 and l_2. The sum $p + q$ is:

   (A) $\frac{15}{2}$ (B) $\frac{17}{2}$ (C) 7

   (D) 8 (E) 9

   Solution

We see that l_1 and l_2 make a square with the origin and (p, q). Using the slopes of parallel/perpendicular lines, the slope of $l_2 = \frac{4}{3}$. Thus l_2 goes up 4 and right 3 units from $(4, -3)$ to the point of intersection, or l_1 goes down 3, right 4 units from $(3, 4)$ to the point of intersection. Therefore, the point of intersection is $(7, 1)$

Answer: D

5. Joshua has chosen a 3-digit number. You are permitted to ask questions to which the answer will be "yes" or "no." The smallest number of questions you must ask Joshua in order to determine his number with certainty is:

(A) 8 or fewer (B) 9 to 11 (C) 12 to 19 (D) 20 to 100 (E) more than 100

Solution

The three digits go from 100 to 999, 899 numbers in total. You can rule out half each time with questions like "Is it below 550?". It it's true, your next question would be: Is it below 325? Otherwise, if it's above 550, your next question would be, "is it below 775?". Each time one reduces the list of possible numbers by one half. Now since

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 900 = \frac{900}{1024} < 1,$$

one will need at most 10 questions to determine the right number.

Answer: B

6. A straight line passes through three points with coordinates $(0, 12)$, $(x, 93)$ and $(100, 120)$. The value of x is:

(A) 69 (B) 70 (C) 72 (D) 75 (E) 77

Solution

$$slope = \frac{120-12}{100-0} = \frac{108}{100} = \frac{93-12}{x}.$$

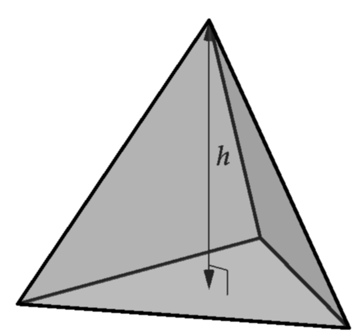
Therefore, $108x = 8100$, $x = \frac{8100}{108} = 75$.

Answer: D

7. All four faces of the tetrahedron shown are equilateral triangles of side length 2 units. The height h of this tetrahedron is:

(A) $\frac{2}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) $\sqrt{\frac{8}{3}}$

(D) $\sqrt{8}$ (E) 3



Solution

The height of one of the triangular faces is $\sqrt{3}$. Drawing a right triangle on a plane through the height of the tetrahedron and an edge will have side lengths h, $\frac{2\sqrt{3}}{3}$, and 2. Thus we have $h^2 + \frac{4}{3} = 4$. Solving for h we get $h = \sqrt{\frac{8}{3}}$.

Answer: C

8. Xavier, Ximena, Xander, Yolanda, Yuri, and Yosef go to math class and sit in a row of 6 seats. The probability that at least one pair whose names start with the same letter sit next to each other is:

(A) 20% (B) 50% (C) 80% (D) 85% (E) 90%

Solution

There are $6! = 720$ ways to sit the six people. There are two arrangements of X's and Y's where same letters do not occur together: $XYXYXYXY$ and $YXYXYX$. There are $3!^2 = 36$ sittings for both. Therefore the probability that no people with same first letter sit together is $\frac{72}{720} = \frac{1}{10}$ So the probability that two people with the same first letter sit together is $\frac{9}{10}$.

Answer: E

9. Suppose x and y are positive integers satisfying $x^2 = y^2 + n$. A value of n that is **not** possible is:

(A) 149 (B) 150 (C) 151 (D) 152 (E) 153

Solution

If n is even and $n = x^2 - y^2 = (x + y)(x - y)$, we have $x + y$ and $x - y$ are both even so n is divisible by 4. We note that 150 is even, but not divisible by 4.

Answer: B

10. The remainder when $1^{2018} + 3^{2018} + 5^{2018} + 7^{2018} + 9^{2018}$ is divided by 20 is:

(A) 5 (B) 7 (C) 13 (D) 15 (E) 17

Solution

Obviously, when divided by 20, the remainder for any power of 1 equals 1. The powers of 3: $3, 9, 27, 81, 243, 729, \ldots$ when divided by 20 have remainders following the pattern $3, 9, 7, 1, 3, 9, 7, 1 \ldots$. Since $2018 \div 4$ has a remainder of 2. So $3^{2018} \div 20$ has a remainder which is second in the pattern, that is, 9. $5^n \div 20$ always has a remainder of 5. $7^n \div 20$ has remainder pattern $7, 9, 3, 1, 7, 9, 3, 1, \ldots$ and $9^n \div 20$ has remainder pattern $9, 1, 9, 1, \ldots$. So $7^{2018} \div 20$ has remainder 9 and $9^{2018} \div 20$ has remainder 1. Adding these remainders, $1 + 9 + 5 + 9 + 1 = 25$. Given that $25 \div 20$ has a remainder of 5, the answer is 5.

Answer: A

11. Each of the letters $A, B, C, D, E, F, G, H, I$, and J is assigned to exactly one digit from 0 through 9, with no two letters being assigned to the same digit. The digits have the property that $ABABFJAI \div ABC = DEFGD$, with no remainder. To the right is the completed long division. The digit assigned to H is:

(A) 1 (B) 3 (C) 5

(D) 7 (E) 9

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                DEFGD
ABC ) ABABFJAI
 (2)  AAAI
      ----
       DIF
 (3)   FIF
       ----
 (1)    AGGJ
         DDB
         ---
          AAA
          GGG
          ----
          AAAI
          AAAI
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Solution

From (1) we see that $A = 1$. By (3), it follows that $B = 2$. Looking at (2) again, we have $10 - A = D$, and hence $D = 9$. Also, $12 - I = I$. Therefore, $I = 6$. Looking at (3), again, we have $9 - F = A = 1$. Therefore, $F = 8$. Also, by (1), it follows that $G = 0$. Going back to (1), $D \times C = I(mod\ 10)$ or $9C = 6(mod\ 10)$. Thus $C = 4$. From (1), we have $J - B = A$. So $J = 2 + 1 = 3$. From (3), we have $E \times C = F(mod\ 10)$, or $4E = 8(mod\ 10)$. Given that $E \neq 2$, it follows that $E = 7$. We have: $A = 1, B = 2, C = 4, D = 9, E = 7, F = 8, G = 0, H = ?, I = 6, J = 3$. It follows that $H = 5$, the last digit available.

Answer: A

12. A girl walks at 4 km/hr, a boy walks at 3 km/hr, and a dog runs at 6 km/hr. The girl and the boy are 2 km apart on a straight road, and the dog is midway between them. The girl follows after the boy who walks away from her, and the dog runs back and forth between the two of them. If the dog starts by running after the boy, then the number of km between the girl and the dog after one hour is:

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 1 (E) $1\frac{1}{3}$

Solution

Place everyone on a number line to begin with (girl is at $x = 0$, dog is at $x = 1$, boy is at $x = 2$). The dog is catching up to the boy at a rate of $3km/hr$, so in $\frac{1}{3}$ of an hour it will have closed the 1 km gap between them. After $\frac{1}{3}$ of an hour (20 minutes), the girl is at $x = \frac{4}{3}$, the dog is at $x = 3$, and the boy is at $x = 3$ and the dog turns back toward the girl. There is $\frac{5}{3}$ km ($3 - \frac{4}{3}$) between them, and they are closing the gap at 10 km/hr. It will take $\frac{\frac{5}{3}}{10} = \frac{1}{6}$ hr (10 minutes) to meet. So at 30 minutes, the boy is at $x = 4.5$, and the girl and the dog are at $x = 2$. Since the dog is 2.5 km behind the boy, he won't catch him (and change direction) in the next half hour. After 1 hour, the girl is at $x = 4$, and the dog is at $x = 5$. They are 1 km apart.

Answer: D