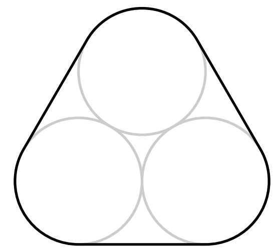
BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2017

Senior Final, Part B Problems & Solutions

1. Three cylindrical pipes of radius 1 m are held together by a tight metal band as shown. The length of the metal band is $a + b\pi$ meters, where a and b are positive integers. Determine a and b.



Solution

The metal band consists of the three congruent arcs on the circles along with the three congruent line segments connecting the circles. The symmetry of the figure tells us the length of the arc joining two tangent points of a circle is one-third the circumference of the circle which is $\frac{1}{3} \cdot 2\pi = \frac{2\pi}{3}$. Since a tangent is perpendicular to the radius at the point of tangency, the length of each line segment is the length of two radii $= 2 \cdot 1 = 2$. The total length of the metal band is then $3(2) + 3 \cdot \frac{2\pi}{3} = 6 + 2\pi$. It follows $a = 6$ and $b = 2$.

Answer: $a = 6, b = 2$

2. Six red balls and k blue balls are randomly placed in a line. The probability that the balls at each end of the line are the same color is $\frac{1}{2}$. Find all possible values of k.

Solution

There is a $6/(k+6)$ probability that the left-most ball is red. If this ball is red, then there is a $5/(k+5)$ probability that the right-most ball is also red.

There is a $k/(k+6)$ probability that the left-most ball is blue. If this ball is red, then there is a $(k-1)/(k+5)$ probability that the right-most ball is also blue.

Therefore,

$$\frac{1}{2} = \frac{6 \cdot 5 + k(k-1)}{(k+6)(k+5)}$$

which simplifies to

$$k^2 - 13k + 30 = (k-3)(k-10) = 0.$$

It follows that $k = 3$ or $k = 10$.

Alternate solution:

There are total of $6 + k$ balls with $C(6+k, 6) = \frac{(6+k)!}{6!k!}$ ways to arrange them. The number of arrangements with red balls in the first and last position is calculated by putting a red ball in those two positions. This leaves 4 red and k blue balls to arrange for a total of

$$C(4+k, 4) = \frac{(4+k)!}{4!k!}$$

arrangements.

Similarly, putting blue in first and last gives

$$C(4+k, 6) = \frac{(4+k)!}{6!(k-2)!}$$

arrangements. Thus,

$$\frac{1}{2} = \frac{\frac{(4+k)!}{4!k!} + \frac{(4+k)!}{6!(k-2)!}}{\frac{(6+k)!}{6!k!}}$$
$$= \frac{6 \cdot 5 \cdot (4+k)! + k \cdot (k-1) \cdot (4+k)!}{(6+k)!}.$$

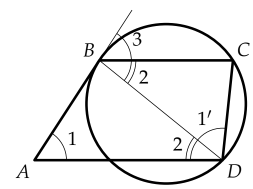
Simplifying further gives

$$60 + 2k(k-1) = (6+k)(5+k) \implies k^2 - 13k + 30 = (k-10)(k-3) = 0.$$

There are either $k = 10$ or $k = 3$ blue balls.

Answer: $k = 3$, $k = 10$

3. A circle passes through the vertices B, C and D of trapezoid $ABCD$. If AB is a tangent to this circle, Prove that $BD = \sqrt{AD \times BC}$.



Solution

Indicating angles as in the drawing, we have $\angle 1 = \angle 3$, because $AD \parallel BC$, and $\angle 1' = \angle 3$, because they are both equal to half of the arc BC (recall AB is tangent to the circle). Hence $\angle 1 = \angle 1'$. Also, $\angle 2 = \angle 2'$, because $AD \parallel BC$, and therefore, the triangles BDC and DAB are similar. But then

$$\frac{BD}{AD} = \frac{BC}{BD},$$

which gives $BD^2 = AD \cdot BC$, so that $BD = \sqrt{AD \times BC}$.

Answer: see proof.

4. The function f is not defined at $x = 0$, but for all non-zero real numbers x the function f satisfies the equation

$$2f(x) + f\left(\frac{1}{x}\right) = 15x.$$

Determine the smallest integer n for which $f(n) > 2017$.

Solution

Substituting $\frac{1}{x}$ for x, we obtain $2f(\frac{1}{x}) + f(x) = \frac{15}{x}$. Multiplying by 2, subtracting from the given equation, solving for $f(\frac{1}{x})$, and re-substituting $\frac{1}{x}$, we find

$$f(x) = \frac{10x^2 - 5}{x} = 10x - \frac{5}{x}.$$

We want the smallest integer n such that $10x - \frac{5}{x} > 2017$. Since x is an integer and $\frac{5}{x}$ decreases as x increases, we can make the estimate $10x \approx 2017$ or $x \approx 201$. Straightforward calculation shows $f(201) < 2017$ while $f(202) > 2017$. It follows that 202 is the smallest integer satisfying the conditions of the problem.

Answer: 202

5. Let a, b, c be real numbers for which $a+b+c=0$, $a^2+b^2+c^2=30$, and $a^3+b^3+c^3=60$. Determine the values of $a^4+b^4+c^4$ and $a^5+b^5+c^5$.

Solution

From $a+b+c=0$, we have $a=-(b+c)$. Substituting this in the second equation, we find

$$a^2+b^2+c^2=(-(b+c))^2+b^2+c^2=2b^2+2c^2+2bc=30$$

and, hence,

$$b^2+c^2+bc=15.$$

Similarly, by substituting $a=-(b+c)$ in the third equation, expanding, regrouping and canceling, we find

$$a^3+b^3+c^3=(-(b+c)^3+b^3=60$$

so

$$b^2c+bc^2=-20.$$

Now we compute the two required sums using the relations just derived. We have

$$\begin{aligned} a^4+b^4+c^4 &= (-(b+c))^4+b^4+c^4 \\ &= 2(b^4+c^4+2b^3c+3b^2c^2+2bc^3) \\ &= 2(b^2+c^2+bc)^2 \\ &= 2\cdot 15^2 = 450 \end{aligned}$$

and

$$\begin{aligned} a^5+b^5+c^5 &= (-(b+c))^5+b^5+c^5 \\ &= -(b^5+5b^4c+10b^3c^2+10b^2c^3+5bc^4+c^5)+b^5+c^5 \\ &= -5(b^4c+2b^3c^2+2b^2c^3+bc^4) \\ &= -5(b^2+c^2+bc)(b^2c+bc^2) \\ &= (-5)(15)(-20) = 1500. \end{aligned}$$

Alternate solution: Let

$$\begin{cases} a+b+c=0 \\ ab+bc+ca=p \\ abc=q \end{cases}$$

Then:

$$\begin{aligned} a^2+b^2+c^2 &= (a+b+c)^2-2(ab+bc+ca) \\ &= 0-2p=-2p \end{aligned}$$

$$\begin{aligned} a^3+b^3+c^3 &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc \\ &= 3q. \end{aligned}$$

$$\begin{aligned} a^4+b^4+c^4 &= (a^2+b^2+c^2)^2-2(a^2b^2+b^2c^2+a^2c^2) \\ &= (a^2+b^2+c^2)^2-2((ab+bc+ca)^2-2(a^2bc+ab^2c+abc^2)) \\ &= (a^2+b^2+c^2)^2-2((ab+bc+ca)^2-2abc(a+b+c)) \\ &= (-2p)^2-2(p^2-0) \\ &= 2p^2. \end{aligned}$$

$$\begin{aligned}
a^5+b^5+c^5 &= (a^4+b^4+c^4)(a+b+c) - (ab^4+ac^4+ba^4+bc^4+ca^4+cb^4) \\
&= 2p^2.0 - (ab^4+ac^4+ba^4+bc^4+ca^4+cb^4) \\
&= -(ab+ac+bc)(a^3+b^3+c^3) + (a^3bc+ab^3c+abc^3) \\
&= -(ab+ac+bc)(a^3+b^3+c^3) + abc(a^2+b^2+c^2) \\
&= -p(3q) + q(-2p) \\
&= -5pq.
\end{aligned}$$

Given that

$$a^2+b^2+c^2 = 30 \quad \text{and} \quad a^3+b^3+c^3 = 60,$$

we have $p = -15$ and $q = 20$, which implies that

$$a^4+b^4+c^4 = 2(-15)^2 = 450$$

and

$$a^5+b^5+c^5 = -5(-15)(20) = 1500.$$

Answer: $a^4+b^4+c^4 = 450$**;** $a^5+b^5+c^5 = 1500$