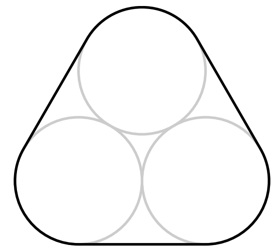
BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2017

Junior Final, Part B Problems & Solutions

1. Three cylindrical pipes of radius 1 m are held together by a tight metal band as shown. The length of the metal band is $a + b\pi$ meters, where a and b are positive integers. Determine a and b.



Solution

The metal band consists of the three congruent arcs on the circles along with the three congruent line segments connecting the circles. The symmetry of the figure tells us the length of the arc joining two tangent points of a circle is one-third the circumference of the circle which is $\frac{1}{3} \cdot 2\pi = \frac{2\pi}{3}$. Since a tangent is perpendicular to the radius at the point of tangency, the length of each line segment is the length of two radii $= 2 \cdot 1 = 2$. The total length of the metal band is then $3(2) + 3 \cdot \frac{2\pi}{3} = 6 + 2\pi$. It follows $a = 6$ and $b = 2$.

Answer: $a = 6$, $b = 2$

2. Fill the following table so that the following three conditions are met:

- Each row contains all of the numbers 1, 2, 3, 4, 5, 6, 7 in some order.
- The sum of the numbers in each column is 12.
- Each column contains three different numbers.

1	2	3	4	5	6	7
				6		
		5				

Solution

Label the grid as follows:

1	2	3	4	5	6	7
A	B	C	D	6	E	F
G	H	5	I	J	K	L

Based on the three conditions, we must have the following:

- Since $5 + 6 + J = 12$ and $3 + C + 5 = 12$, we must have $C = 4$ and $J = 1$.
- Since $6 + E + K = 12$, we must have (E, K) equal one of $(1, 5), (2, 4), (4, 2), (5, 1)$. Since 1 and 5 already appear in the third row, and 4 appears in the second row, we must have $E = 2$ and $K = 4$.
- Since $7 + F + L = 12$, we must have (F, L) equal one of $(1, 4), (2, 3), (3, 2), (4, 1)$, from which we quickly see that $L = 2$ and $F = 3$.
- One of A, B, D must be 1. If $A = 1$ or $B = 1$, then G or H exceeds 7, which is impossible. Thus, we must have $D = 1$, from which we get $I = 7$.

- One of A, B must be 5. If $B = 5$, then $H = 5$, which violates the third condition. Thus, $A = 5$, from which we get $G = 6$, $B = 7$, and $H = 3$.

The completed grid is

1	2	3	4	5	6	7
5	7	4	1	6	2	3
6	3	5	7	1	4	2

Answer: see solution

3. Let x and y be two numbers with a difference of 4 and a product of 1. Determine the value of $\frac{x}{y} + \frac{y}{x}$.

Solution

We are given $x - y = 4$ and $xy = 1$. Solving for y in the second equation, substituting the result in the first equation, multiplying by x and re-arranging we arrive at the quadratic equation $x^2 - 4x - 1 = 0$. By the quadratic formula, we have

$$x = \frac{4 \pm \sqrt{(-4)^2 + 4}}{2} = 2 \pm \sqrt{5}$$

If $x = 2 + \sqrt{5}$, then $y = x - 4 = -2 + \sqrt{5}$ in which case

$$\frac{x}{y} + \frac{y}{x} = \frac{2+\sqrt{5}}{-2+\sqrt{5}} + \frac{-2+\sqrt{5}}{2+\sqrt{5}} = 18$$

The other value of x leads to the same result.

Alternate solution:

We have

$$(x - y)^2 = 4^2 = 16$$

but also

$$(x - y)^2 = x^2 + y^2 - 2xy,$$

so

$$x^2 + y^2 - 2xy = 16 \implies x^2 + y^2 = 16 + 2xy.$$

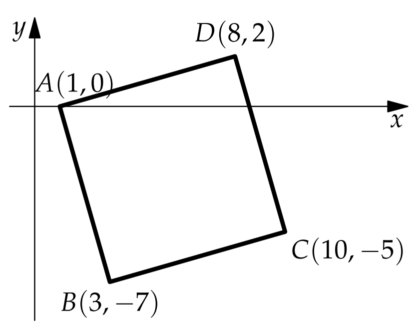
Since $xy = 1$ is given, we have $x^2 + y^2 = 18$. Now:

$$\frac{x}{y} + \frac{y}{x} = \frac{x^2 + y^2}{xy} = \frac{18}{1} = 18.$$

Answer: 18

4. The x-coordinates of the vertices of a square in the plane are 1, 3, 8 and 10. Determine the area of the square.

Solution



Let $A = (1, 0)$, $B = (3, b)$, $C = (8, c)$ and $D = (10, d)$. These four points form a square, in some order. Since the x-coordinates of A and D add up to the x-coordinates of B and C, the only possibility is that AD and BC are diagonals, i.e., AB and AC are perpendicular. (We can also see this from drawing a diagram, that AB and AC must be perpendicular to each other).

AB has run $3 - 1 = 2$ and rise $b - 0 = b$, for a slope of $b/2$.

AC has run $8 - 1 = 7$ and rise $c - 0 = c$, for a slope of $c/7$.

Since $AB = AC$ and AB is perpendicular to BC, we quickly see that we must have either $(b, c) = (7, -2)$ or $(b, c) = (-7, 2)$.

(For a more formal proof, we see that these two statements imply $2^2 + b^2 = 7^2 + c^2$, and $(b/2) \cdot (c/7) = -1$, from which we get the two solutions for (b, c) above.)

The area of the square must be

$$AB \cdot AC = AB^2 = 2^2 + b^2 = 53.$$

Answer: 53

5. Let N be a 3-digit number with three distinct non-zero digits. We say that N is mediocre if it has the property that when all six 3-digit permutations of N are written down, the average is N. For example, $N = 481$ is mediocre, since the average of $\{481, 148, 184, 418, 814, 841\}$ is 481. Determine the largest three-digit mediocre number.

Solution

Suppose $M = abc = 100a + 10b + c$ is a mediocre number. The six permutations of a, b and c are

$$abc, acb, bac, bca, cab, cba.$$

If we form the required six numbers from the permutations, sum them, and divide by 6 to get the average, then the condition satisfied by M translates to

$$100a + 10b + c = \frac{100(2a + 2b + 2c) + 10(2a + 2b + 2c) + (2a + 2b + 2c)}{6} = \frac{222(a + b + c)}{6}$$

which simplifies to

$$7a = 3b + 4c.$$

We must find distinct values of a, b, and c such this equation is satisfied and the number $N = 100a + 10b + c$ is maximized. The equation $7a = 3b + 4c$ can be rewritten as $3(a - b) = 4(c - a)$. This implies $a - b = 4k$ and $c - a = 3k$, for some non-zero k. Therefore, $(a, b, c) = (a, a - 4k, a + 3k)$.

Because we require a,b,c to be digits (i.e. integers between 1 and 9), we must have $k = -1$ or $k = 1$, since $c - b = 7k$.

- If $k = -1$, then $(a, b, c) = (a, a + 4, a - 3)$ implies $a \leq 5$. Thus, $(a, b, c) = (5, 9, 2)$ is the solution that maximizes N.
- If $k = 1$ then $(a, b, c) = (a, a - 4, a + 3)$ implies $a \leq 6$. Thus, $(a, b, c) = (6, 2, 9)$ is the solution that maximizes N.

The answer is $N = 629$.

Answer: 629