BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2016

Senior Final, Part A Problems & Solutions

1. If $10^x = m$, then 10^{x+2} is equal to:

   (A) m^2 (B) $m + 100$ (C) $100m$ (D) $m + 2$ (E) $2m$

   Solution

   $$10^{x+2} = 10^x 10^2 = 10^x 100 = 100m$$

   Answer: C

2. There are five married couples in a room. Each person shakes hands with everyone else, but does not shake hands with their own partner. How many handshakes happen?

   (A) 10 (B) 15 (C) 24 (D) 40 (E) 45

   Solution
   10 people would have 45 handshakes (the first person shakes hands with 9 people, the second has 8 left to shake with, the third has 7 left. . .)

   $$9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = (9)(10)/2 = 45.$$

   But the five married pairs do not shake hands, so the answer is $45 - 5 = 40$.

   Answer: D

3. A series of books was published at seven-year intervals. When the seventh book was issued, the sum of the publication years was 13, 524. When was the first book published?

   (A) 1911 (B) 1918 (C) 1932 (D) 1939 (E) 1953

   Solution
   Let n be the publication the year of the first book. The books were published at years n, $n + 7$, $n + 14$, $n + 21$, $n + 284$, $n + 35$, and $n + 42$. The sum is $7n + 147 = 13,524$. Solving for n gives $n = 1911$.

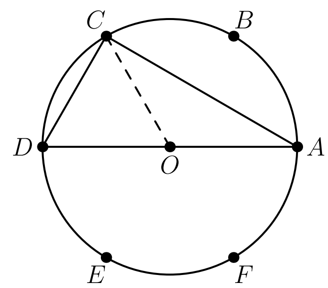
   Answer: A

4. Six points are equally spaced around a circle of radius 6. Three of the points are vertices of a triangle that is neither equilateral nor isosceles. The area of the triangle is:

   (A) $12\sqrt{3}$ (B) $18\sqrt{3}$ (C) 36 (D) $24\sqrt{3}$ (E) 48

   Solution

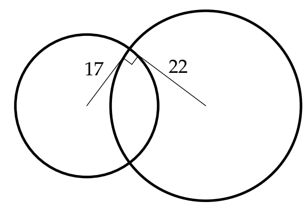
In the diagram the points A, B, C, D, E, and F are equally spaced around the circle. Triangles ACE and BDF are equilateral and any one of the six triangles congruent to ABC are isosceles. The other 12 triangles are neither equilateral nor isosceles and are congruent to triangle ACD shown. Since DA is a diameter of the circle, triangle ACD is a right triangle with right angle opposite the diameter. Further, if O is the centre of the circle, then $\angle COD = 60°$, and, since $DO = CO = 6$, triangle CDO is equilateral with $CD = 6$. Hence, $\angle CDO = 60°$, so that ACD is a $30°$-$60°$-$90°$ triangle with hypotenuse $AD = 12$. Therefore, $CA = 6\sqrt{3}$ and the area of triangle ACD is



$$\text{Area} = \frac{1}{2} \times 6 \times 6\sqrt{3} = 18\sqrt{3}$$

Answer: B

5. A circle of radius 17 intersects another circle, radius 22, at right angles as shown. What is the difference of the areas of the non-overlapping portions?



(A) 115π (B) 135π (C) 155π

(D) 175π (E) 195π

Solution

The area of the larger circle is $22^2\pi$; the area of the smaller circle is $17^2\pi$. Suppose x is the area of the overlapping portion. Then the difference of the areas of the non-overlapping portions is

$$\begin{aligned}(22^2\pi - x) - (17^2\pi - x) &= (22^2 - 17^2)\pi \\ &= (22+17)(22-17)\pi \\ &= (39)(5)\pi = 195\pi.\end{aligned}$$

Answer: E

6. Professor Paul gave course marks based on an average of a series of tests. Before Suzanne's last test she realized that she would have to score 97% in order to average 90% for the course. On the other hand, if she scored as low as 73% she would still be able to average 87%. How many tests were in Professor Paul's series?

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Solution

Let the sum of her marks up to now be S and the number of tests be N. In the first case, her calculation of her average is

$$\frac{S+97}{N} = 90$$

and her second calculation is

$$\frac{S+73}{N} = 87.$$

Thus

$$S = 90N - 97 = 87N - 73 \implies 3N = 24 \implies N = 8.$$

Answer: B

7. For any positive integer n, define $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. Consider the product

$$\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{2n-1}{2n}\right)$$

where n is a positive integer. A general formula for this product for any positive integer n is:

(A) $\dfrac{(2n)!}{2^{2n}(n!)^2}$ (B) $\dfrac{2n!}{2^{n+1}(n^2)!}$ (C) $\dfrac{(2n)!}{2^n(n!)^2}$

(D) $\dfrac{2n!}{2^{2n}(n!)^2}$ (E) $\dfrac{(2n)!}{2^{n+1}n!}$

Solution

Combining the fractions gives

$$\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{2n-1}{2n}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

Multiplying numerator and denominator by $2 \cdot 4 \cdot 6 \cdots (2n)$ gives

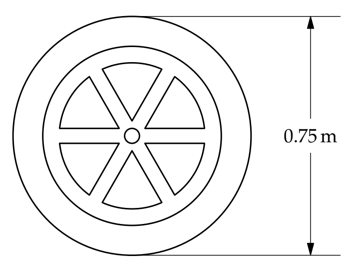
$$\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{2n-1}{2n}\right) = \frac{1 \cdot 2 \cdot 3 \cdots (2n)}{(2 \cdot 4 \cdot 6 \cdots (2n))^2}$$

Then noting that $2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!$ gives

$$\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{2n-1}{2n}\right) = \frac{1 \cdot 2 \cdot 3 \cdots (2n)}{(2^n n!)^2} = \frac{(2n)!}{2^{2n}(n!)^2}$$

Answer: A

8. The wheels of a car have six-way radial symmetry (see diagram) and measure 0.75 m in diameter. A video camera which captures 12 frames per second creates a video of the car in motion. In the video, the wheels of the car appear to be not turning at all. If it is known that the car is traveling at a speed between 15 and 20 m/s, which of the following is a possible speed for the car, measured in m/s?



(A) 4π (B) 15 (C) 18

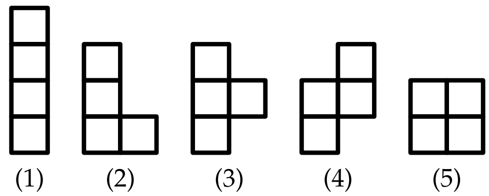
(D) 6π (E) 24

Solution

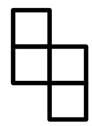
For the wheels to appear stationary, they must be turning some multiple (n) of 60° every one-twelfth of a second, so they're turning at $12n/6 = 2n$ revolutions per second. Since their diameter is 0.75 m, the car is moving at $1.5n\pi$ m/s. This is roughly $1.5n\pi \approx 4.7n$, which only falls between 15 and 20 if $n = 4$. Thus the car is traveling at 6π m/s.

Answer: D

9. A tetromino is a geometric shape formed by adjoining four squares with one another, edge to edge. There are five distinct tetrominoes:



Shapes that can be obtained by rotating or reflecting one of these five tetrominoes are not considered distinct tetrominoes. For example, the shape



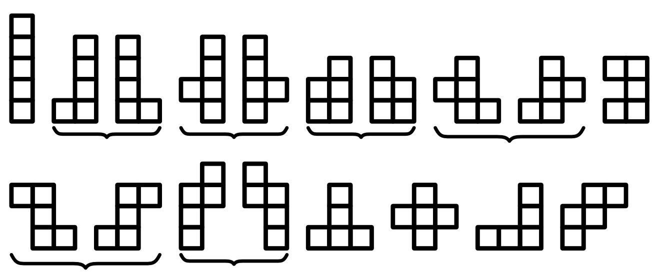
is considered to be the same as (4) above because it is a reflection of (4).

A pentomino is formed by adjoining five squares with one another, edge to edge. How many distinct pentominoes are there?

(A) 10 (B) 12 (C) 14 (D) 16 (E) 18

Solution

There are 18 distinct pentominoes, as shown below.



For six pairs of these one of the pentominoes can be obtained by a reflection of the other, but not by a rotation. These pairs are shown grouped together. Hence, there are 12 distinct pentominoes if reflections are not counted as distinct.

Answer: B

10. 70% of the adult males of a community have brown eyes, 75% have dark hair, 85% are over six feet tall, and 90% weigh more than 140 pounds. What is the minimum percentage of people who must have all four characteristics?

(A) 0% (B) 5% (C) 10% (D) 15% (E) 20%

Solution

For brown eyes and dark hair , trying to leave them separate, we see $70\% + 75\% = 145\%$, which is too high (over 100%), so we must have at least 45% overlap. For the overlap and over 6 ft tall, trying to leave them separate, we see $45\% + 85\% = 130\%$, which is too high, so we must have at least 30% overlap. For the new overlap and weighing over 140 pounds, trying to leave them separate, we see $30\% + 90\% = 120\%$, which is too high, so we must have at least 20% overlap.

Answer: E