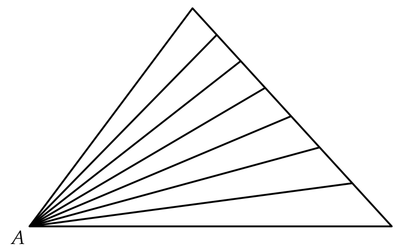
BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2016

Junior Final, Part B Problems & Solutions

1. How many triangles appear in the diagram?



Solution

There are 7 single skinny triangles, 6 "doubles," 5 "triples," 4 "four-thick," 3 "five-thick," 2 "six-thick," and the whole triangle. Answer: 7+6+5+4+3+2+1=28.

Alternative Solution:

Each triangle must have A as one of its vertices. Two points on the edge opposite A form the other two vertices. There are

$$C(8,2) = \frac{8!}{2!6!} = 28$$

distinct choices for the vertices opposite A, hence there are 28 distinct triangles.

Answer: 28

2. What is the smallest k such that the sum

$$1 + 11 + 111 + 1111 + 11111 + \cdots + \underbrace{1111\cdots11}_{k\text{ 1's}}$$

is divisible by 9?

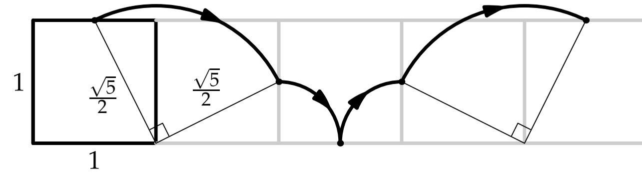
Solution

We know that a number is divisible by 9 if and only if the sum of the digits is 9. We also know that first 9 sums are: 1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, with the respective sums of digits: 1, 3, 6, 10, 15, 21, 28, 36, 45. The first sum of digits to be divisible by 9 is $1+2+3+4+5+6+7+8 = 36$, which happens when $k = 8$. Answer: $k = 8$

Answer: 8

3. A 1 cm cube has a dot at the centre of the top face. The cube is rolled forward until the dot is again on top. What is total length of the path traced in space by the dot?

Solution



During each of the four rolls of the cube, the path of the dot is a quarter of a circle (when viewed from the side). Since the cube has side length one, the first and fourth circles have radius $\frac{\sqrt{5}}{2}$ and the second and third circles have radius 0.5. Thus, the dot travels a distance of

$$\frac{1}{4}\left(\sqrt{5}\pi + \pi + \pi + \sqrt{5}\pi\right) = \frac{\pi(1+\sqrt{5})}{2}.$$

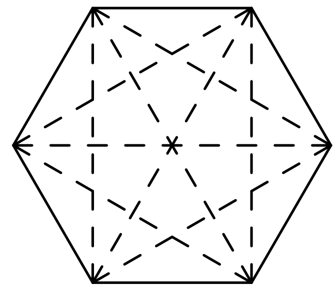
Answer: $\frac{\pi}{2}(1+\sqrt{5})$

4. For a regular polygon with n sides ($n > 3$), a *diagonal* is a line segment joining any two non-adjacent vertices.

   (a) Find the number of diagonals in a regular hexagon.

   Solution

   See diagram. Answer: 9.



   (b) Find a formula for the number of diagonals in a regular polygon with n sides.

   Solution

   From each vertex, you can go to any vertex except itself or its two adjacent neighbors. So we have n ways to choose the first vertex, times $n-3$ ways to choose the second vertex. Now we divide by two because line segment BA is the same as line segment AB, so we've counted each diagonal twice. Answer: $n(n-3)/2$.

   Answer: $n(n-3)/2$

   (c) Find all the regular polygons with eight times as many diagonals as sides.

   Solution

   The number of diagonals is

$$8n = \frac{n(n-3)}{2} \implies 16n = n^2 - 3n \implies n(n-19) = 0$$

   so n must be 19.

   Answer: 19

5. We have 8 boxes – one red, the others blue and yellow – each containing a different number of balls: 11, 14, 19, 23, 29, 32, 41, and 46. The total number of balls in the yellow boxes is twice the total number of balls in the blue boxes. Determine the number of balls in the red box.

   Solution

   The total number of balls is 215. We can make a table with every possible number of balls in the red box, and eliminate those that give a contradiction:

# in red box	# remaining	# in blue boxes (=remaining/3)	possible cases
11	204	68	none
14	201	67	none
19	196	not possible (196 not divisible by 3)	
23	192	64	(23,41) not possible (have 23 in red box)
29	186	62	(11,19,32)
32	183	61	(29,32) not possible (have 32 in red box)
41	174	58	none
46	169	not possible (169 not divisible by 3)	

Checking possible cases is a little tedious, but is a little quicker if you consider the numbers mod 3. So, the only possible choice is 29 in the red box.

Answer: 29