# BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2006

## Solutions

## Junior Preliminary

1. Let $s$ be the side length of the one of the squares. The perimeter of the rectangle $PQRS$ is $8s = 120 \Rightarrow s = 15$. Thus, the area of one of the squares is $s^2 = 225\,\text{cm}$.

**Answer is (C).**

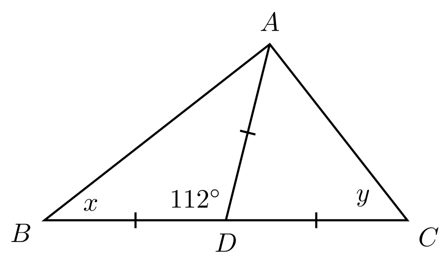
---

2. The frequency of Efran's visits must be a divisor of 84 that is larger than seven. The divisors of 84 are: 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84. If Efran visits every 14, 21, or 42 days, then all four of the friends would show up every 42 days. Since 18 is not a divisor of 84, of the choices given the only choice is that Efran visits every 12 days. Note that 28 or 84 days are also possible.

**Answer is (A).**

---

3. Consider the diagram




Triangles $ABD$ and $ADC$ are both isosceles. It follows that the angles $x$ and $y$ must satisfy the equations $2x + 112 = 180$ and $2y + (180 - 112) = 180$. It follows that $x = 34°$ and $y = 56°$. The difference in these angles is $y - x = 22°$.

**Answer is (D).**

---

4. If $10^{100} = 10000^n$, then using the laws of exponents

$$10000^n = \left(10^4\right)^n = 10^{4n} = 10^{100}$$

so that $4n = 100 \Rightarrow n = 25$.

**Answer is (B).**

---

5. Let $b$ be the mass of one block, $m$ the mass of one marble, and $t$ the mass of one top. Then we have the following two equations for $b$, $m$, and $t$:

$$3b + t = 15m \text{ and } t = b + 7m$$

Substituting $t$ from the second equation into the first gives $3b + b + 7m = 15m \Rightarrow 4b = 8m \Rightarrow b = 2m$. Substituting this into the second equation gives $t = 2m + 7m = 9m$. Thus, it takes 9 marbles to balance one top.

**Answer is (C).**

6. If the prime factorization of the whole number $x$ is $x = p^\alpha q^\beta r^\gamma$, then the number of divisors $x$ has is $(\alpha+1)(\beta+1)(\gamma+1)$. If $x$ has exactly 12 divisors, this product must equal 12. So we must factor 12 into a product of whole number factors. The possible factorings are

$$12 = 12 \cdot 1 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$$

Consider each possible factoring:

i) For the first factoring there is only one prime factor with $\alpha = 11$ for $p = 2$, giving $x = 2^{11} = 2048$.

ii) For the second there are two prime factors with $\alpha = 5$ and $\beta = 1$ for $p = 2$ and $q = 3$, giving $x = 2^5 \cdot 3^1 = 96$.

iii) For the third there are two prime factors with $\alpha = 3$ and $\beta = 2$ for $p = 2$ and $q = 3$, giving $x = 2^3 \cdot 3^2 = 72$.

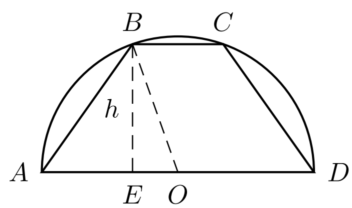
iv) For the fourth there are three prime factors with $\alpha = 2$, $\beta = 1$, and $\gamma = 1$ for $p = 2$, $q = 3$, and $r = 5$ giving $x = 2^2 \cdot 3^1 \cdot 5^1 = 60$.

The 60 is the smallest whole number that has exactly 12 distinct divisors, which is in the interval $55 \le x \le 65$.

**Answer is (B).**

---

7. Consider the diagram below




Let $a = \overline{AD}$, $b = \overline{BC}$ and $h$ be the perpendicular distance between the chord $BC$ and the diameter $AD$. The area $A$ of the trapezoid $ABCD$ is given by

$$A = \frac{1}{2}(a+b)h.$$

Since $a = d$ and $b = d/3$, where $d$ is the diameter of the circle, we have

$$A = \frac{1}{2}\,\frac{4d}{3}\,h = \frac{2}{3}\,dh\,.$$

We need only determine $h$. Consider the right triangle $BOE$ formed by dropping a perpendicular from $B$ to the point $E$ on the diameter $AD$. The length of this perpendicular is $h$, and $\overline{BO} = d/2$ since $BO$ is a radius of the circle. Further, by symmetry $2\overline{AE} + \overline{BC} = d \Rightarrow \overline{AE} = d/3$, so that $\overline{EO} = d/2 - d/3 = d/6$. Then using Pythagoras' Theorem gives

$$\frac{d^2}{36} + h^2 = \frac{d^2}{4},$$

which gives $h = \dfrac{\sqrt{2}}{3}\,d$. It follows that the area of the trapezoid is given by

$$A = \frac{2\sqrt{2}}{9}\,d^2 = \frac{2\sqrt{2}}{9}\,36 = 8\sqrt{2}\,.$$

If you don't remember the formula for the area of a trapezoid, we can break the area into the sum of the areas of two triangles and a rectangle. Then we get

$$A = \frac{1}{2}\,\frac{d}{3}\,h + \frac{d}{3}\,h + \frac{1}{2}\,\frac{d}{6}\,h = \frac{d}{3}\,h + \frac{d}{3}\,h = \frac{2}{3}\,dh$$

as above.

**Answer is (E).**

8. Recall that $\sqrt{x}\sqrt{y} = \sqrt{xy}$, so that

$$\left(\sqrt{1+\frac{\sqrt{3}}{2}}\right)\left(\sqrt{1-\frac{\sqrt{3}}{2}}\right) = \sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{1-\frac{3}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

**Answer is (C).**

---

9. We want to write $57.24\overline{5724}\%$ as a rational number in lowest terms. Since

$$\begin{aligned} 57.24\overline{5724}\% &= 0.5724 + 0.00005724 + 0.000000005724 + \ldots \\ &= 0.5724\left(1 + \frac{1}{10^4} + \left(\frac{1}{10^4}\right)^4 + \ldots\right) \\ &= 0.5724\frac{1}{1-10^{-4}} = 0.5724\frac{10000}{9999}\,, \end{aligned}$$

it follows that

$$57.24\overline{5724}\% = \frac{5724}{9999} = \frac{636}{1111}\,.$$

Hence the minimum number of people that could have been asked is 1111.

**Answer is (E).**

---

10. The area that can be grazed is the sector of a circle. Since the triangle is equilateral, the central angle of the sector is $\theta = \frac{\pi}{3}$ radians. The area of this sector must be $\frac{1}{2}A$ so

$$\frac{A}{2} = \frac{\theta}{2}r^2,$$

where $r$ is the radius of the circle. If follows that

$$r = \sqrt{\frac{A}{\theta}} = \sqrt{\frac{3A}{\pi}}\,\text{m}\,.$$

If you are not familiar with radian measure the angle determining the sector is $60^\circ$, since all of the angles in an equilateral triangle are $60^\circ$. Since a sector with a central angle of $60^\circ$ is one sixth of the circle, if the length of the tether is $r$, the radius of the sector, then

$$\frac{1}{6}\left(\pi r^2\right) = \frac{1}{2}A \Rightarrow r^2 = \frac{3A}{\pi}$$

Giving the same result as above.

**Answer is (A).**

11. We only need to consider two initial moves: up to the centre 0 in the second row, or diagonally up and to the left to the left most 0 in the second row. Then by symmetry we just need to multiply by 4. After moving up we can move right or left to one of the other 0's on the second row, or diagonally down to the centre 0 in the second or fourth column. From either of the 0's in the second row there are 5 possible moves. For example, from the left most 0 they are: down left, left, up left, up, and up right. From either of the centre 0's there are 3 possible moves. For example, from the centre 0 in the second column they are: down left, left, and up left. This gives a total of 16 distinct paths with initial move up to the centre 0 in the second row. After moving diagonally up and to the left, there are two possible moves: down to the centre 0 in the second column or right to the centre 0 in the second row. In either case, there are three possible moves, as described above. This gives a 6 more possible paths with initial move up to the centre 0 in the second row, for a total of 22 distinct paths. So the total number of distinct paths is 88.

**Answer is (D).**

---

12. There are

$$\binom{6}{2} = \frac{6 \times 5}{2} = 15$$

distinct pairings of teams. Since each team must play every other four times, the total number of games is $4 \times 15 = 60$.

**Answer is (B).**

---

## Senior Preliminary

1. See Junior Preliminary Problem 9.

---

2. Let $v$ be the initial speed. Recalling that an object moving with speed $v$ for time $t$ goes a distance $d = vt$, and using the given information, we have

$$v(8) - (2v)(3) + (0.4v)(1) = 8v - 6v + 0.4v = 2.4v = 9.6 \Rightarrow v = 4$$

So the initial speed is $4\,\text{m/s}$.

**Answer is (B).**

---

3. Let $x = 123456789$, then the expression in the problem is

$$x^2 - (x+5)(x-5) = x^2 - \left(x^2 - 25\right) = x^2 - x^2 + 25 = 25$$

**Answer is (A).**

---

4. The five vertical cuts divide the block of cheese into twelve pieces. The first of the diagonal cuts passes through a single row of pieces and hence adds three pieces to the total. The second diagonal cut passes through two rows of pieces and adds six pieces to the total. Therefore there are $12 + 3 + 6 = 21$ pieces of cheese.

**Answer is (A).**

5. See Junior Preliminary Problem 7.

---

6. Let $D$ denote this difference, then

$$D = \sum_{k=1}^{100} 3k - \sum_{k=1}^{100} 2k = \sum_{k=1}^{100} k\,.$$

Now, since $\sum_{k=1}^{n} k = n(n+1)/2$,

$$D = \frac{100 \times 101}{2} = 5050\,.$$

**Answer is (B).**

---

7. The first cut always produces two pieces of pie. If the remaining five cuts pass through the centre of the pie, then each cut passes through exactly two pieces. This means that each cut adds two more pieces of pie for a total of twelve pieces of pie. However, if we have $k-1$ lines in the plane, we can always add a $k^{\text{th}}$ line $l$ such that:

   1. $l$ intersects each of the $k-1$ lines, and
   2. $l$ intersects no more than two lines intersect at a single point.

   Moreover, if the $k-1$ lines divide the plane into a total of $m$ regions, then line $l$ adds $k+1$ regions to the total. If we ensure that all intersection points are in the interior of the pie, then the maximum number of pieces $M$ that can be cut from the pie is

$$M = 2 + 2 + 3 + 4 + 5 + 6 = 22\,.$$

**Answer is (E).**

---

8. Suppose that $S$ has $n$ elements, then $S$ has a total of $2^n$ subsets. Since $\chi = 2^n - 24$ is the total number of subsets of the set $T$, $\chi$ must be a power of 2. Since $2^5 - 24 = 8 = 2^3$, the set $S$ must have 5 elements.

**Answer is (D).**

---

9. Every number in the given set of numbers is a multiple of 3 except for 25 and 19. Further, subtracting either 25 or 19 from 50 gives a result that is not a multiple of three. So both 25 and 19 must be included in the sum. Since $25 + 19 = 44$, the other number is 6. This is the only selection of numbers from the given set of numbers that adds to 50. So there are three numbers in the selection.

**Answer is (A).**

---

10. We have $x = \sqrt{2+x}$, which implies that $x$ must satisfy the quadratic equation $x^2 - x - 2 = 0$. Since $x^2 - x - 2 = (x-2)(x+1)$, the solutions of this quadratic equation are $x_{1,2} = 2, -1$. However $x > 0$ so $x = 2$.

**Answer is (A).**

---

11. If we think of the towns as nodes in a graph, then the roads are the edges. The degree of a node is the number of edges connected to it and since each edge is connected to two nodes, the sum of the degrees of all of the nodes in the graph is twice the number of edges. Let $n$ be the number of towns that can be reached by four roads (i.e. the number of nodes of degree 4). Then we find that $3 \times 6 + 4n = 2 \times 21$ or $4n = 24$. There are $6 + 24/4 = 12$ towns in total.

**Answer is (D).**

12. Since the new denominator must be 3 times the new numerator, the new numerator must be a multiple of 3. Since no multiple of 3 ends in 0, the 0 must be swapped into the denominator and it cannot be put into the left most position. Further, it cannot be swapped with the 5, since then there would have to be a 5 in the right most position in the new numerator and then there would also be a 5 in the right most position of the new denominator. Thus, we must swap the 0 in the numerator with the second 4 in the denominator. (The new denominator cannot start with 0). We cannot swap the 1 in the numerator with any of the remaining digits in the denominator, otherwise 3 times the new numerator will be too large. Also, swapping the 1 into the denominator will mean that the new denominator will not be a multiple of 3. This means that there are only four possibilities to try: swap the 6 and the 4, swap the 6 and the 5, swap the 3 and the 4, or swap the 3 and the 5. Only the first and last of these leaves the denominator a multiple of 3. Now we just check the two remaining possibilities:

$$3 \times 1654 = 4962 \neq 4302 \quad \text{and} \quad 3 \times 1534 = 4602$$

So the new numerator is 1534 and the sum of the digits is $1 + 5 + 3 + 4 = 13$.

**Answer is (D).**

---

# Junior Final, Part A

1. Let $w$, $x$, $y$, and $z$ represent the number of tins collected by the four schools, from smallest to largest. Then

$$\begin{aligned} w + x + y + z &= 8888 \\ w + 888 &= z \\ x + 88 &= z \\ y + 8 &= z \end{aligned}$$

$$\begin{aligned} (z - 888) + (z - 88) + (z - 8) + z &= 8888 \\ 4z - 984 &= 8888 \\ z &= \frac{9872}{4} = 2468 \end{aligned}$$

The schools collected 1580, 2380, 2460, 2468. Thus, the second most number of tins collected is 2460.

**Answer is (A).**

---

2. We require $x = 3n + 2$ so $y = 31 - 2n$ where $0 \leq n \leq 15$ for a total of 16 solutions.

**Answer is (B).**

---

3. See the table at the right. There are 36 entries of which 24 are between 16 and 24 inclusive. Hence the required probability is

$$\frac{24}{36} = \frac{2}{3}$$

| + | 5 | 7 | 9 | 11 | 13 | 15 |
|---|---|---|---|---|---|---|
| 5 | 10 | 12 | 14 | 16 | 18 | 20 |
| 7 | 12 | 14 | 16 | 18 | 20 | 22 |
| 9 | 14 | 16 | 18 | 20 | 22 | 24 |
| 11 | 16 | 18 | 20 | 22 | 24 | 26 |
| 13 | 18 | 20 | 22 | 24 | 26 | 28 |
| 15 | 20 | 22 | 24 | 26 | 28 | 30 |

**Answer is (B).**

4. If all of the operations are $\times$, no matter where parentheses are put the result is

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$$

Making the first operation a $+$, with the 1 and 2 grouped together using parentheses to make a sum of 3, and the rest $\times$ gives the largest possible value as

$$(1 + 2) \times 3 \times 4 \times 5 \times 6 = 1080$$

**Answer is (E).**

---

5. See the table at the right. There are 37 entries between 1 and 100 in the table, of which 50, 65, and 85 are repeated; there are 34 distinct entries.

| | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 5 | 10 | 17 | 26 | 37 | 50 | 65 | 82 | |
| 4 | | 8 | 13 | 20 | 29 | 40 | 53 | 68 | 85 | |
| 9 | | | 18 | 25 | 34 | 45 | 58 | 73 | 90 | |
| 16 | | | | 32 | 41 | 52 | 65 | 80 | 97 | |
| 25 | | | | | 50 | 61 | 74 | 89 | | |
| 36 | | | | | | 72 | 85 | | | |
| 49 | | | | | | | 98 | | | |

**Answer is (D).**

---

6. Let $g$ be the number of guests, $r$ be the number of rice bowls, $s$ the number of soup bowls, and $m$ the number of meat dishes. Then we have the following equations

$$2r = 3s = 4m = g$$
$$r + s + m = 65$$

Expressing $r$, $s$, and $m$ in terms of $g$ and substituting in the second equation above gives:

$$\frac{g}{2} + \frac{g}{3} + \frac{g}{4} = 65 \Rightarrow 6g + 4g + 3g = 780 \Rightarrow 13g = 780$$

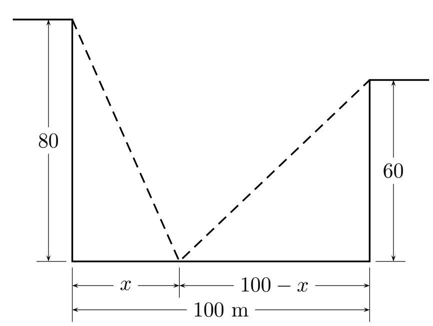
Thus, $g = 60$.

**Answer is (B).**

---

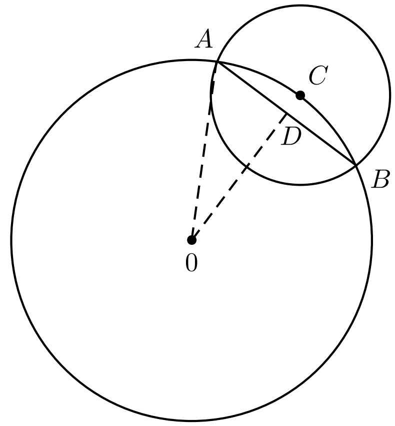
7. Let $x$ be the distance from the pile of grain to the foot of the cathedral tower. See the diagram at the right. Then

$$x^2 + 80^2 = (100 - x)^2 + 60^2 = 10000 - 200x + x^2 + 3600$$
$$200x = 13600 - 6400$$
$$x = 36$$




**Answer is (C).**

8. Consider the diagram below




Let the centre of the larger circle be $O$, the centre of the smaller circle be $C$ and the intersection of $AB$ and $OC$ be $D$. See the diagram above. Let $x = OD$ and $h = DA$. Then

$$x^2 + h^2 = 4 \quad \text{and} \quad (2-x)^2 + h^2 = 1$$

subtracting the two equations gives

$$4x - 4 = 3 \Rightarrow x = \frac{7}{4}$$

solving for $h$ gives

$$h = \sqrt{4 - \frac{49}{16}} = \frac{\sqrt{15}}{4}$$

The chord is

$$2h = \frac{\sqrt{15}}{2}$$

**Answer is (A).**

---

9. With

$$x = \sqrt{1 + \frac{\sqrt{3}}{2}} + \sqrt{1 - \frac{\sqrt{3}}{2}}$$

$$x^2 = 1 + \frac{\sqrt{3}}{2} + 2\sqrt{\left(1 + \frac{\sqrt{3}}{2}\right)\left(1 - \frac{\sqrt{3}}{2}\right)} + 1 - \frac{\sqrt{3}}{2} = 2 + 2\sqrt{1 - \frac{3}{4}} = 3$$

then, since $x > 0$,

$$x = \sqrt{3}$$

**Answer is (E).**

10. The numbers 7, 11, and 13 cannot be used in any of the circles, otherwise one or two of the products would be a multiple of one of these primes, but the other product could not have that prime as a factor. Further, note that the product of the triangle products is a perfect cube, since the three triangle products are equal.

    Now the product of the seven numbers that can be used in the circles has a prime factorization of $\left(2^9\right)\left(3^4\right)\left(5^2\right)$. The product of triangle products is a multiple of this number, since the numbers in circles $E$ and $F$ occur in two of the triangle products. To make this prime factorization a perfect cube we must multiply by $9 = 3^2$ and 5. Hence, 9 and either 5 or 10 must be in circles $E$ and $F$. So the prime factorization of the product of triangle products is $\left(2^9\right)\left(3^6\right)\left(5^3\right)$, and each triangle product is the cube root $\left(2^3\right)\left(3^2\right)(5) = 360$.

    If 9 goes in circle $E$ and 10 in circle $F$, then 4 must go in circle $B$ to make a product of 360 for triangle $DEF$, and 5 (the 5 cannot go with the 10) and 8 must go in circles $A$ and $D$ to make a product of 360 for triangle $ADE$. This leaves 6 and 12 for circles $C$ and $G$, which gives a product of 720 for the triangle $CFG$.

    If 9 goes in circle $E$ and 5 in circle $F$, then 8 must go in circle $B$ to make a product of 360 for triangle $DEF$, and 10 (the 10 cannot go with the 5) and 4 must go in circles $A$ and $D$ to make a product of 360 for triangle $ADE$. This leaves 6 and 12 for circles $C$ and $G$, which now gives a product of 360 for the triangle $CFG$. Thus, 8 must go in circle $B$.

**Answer is (C).**

---

---

## Junior Final, Part B

1. If $s$ is the side length of an equilateral triangle, then the height of the triangle is $h = \frac{\sqrt{3}}{2}s$. Thus the area of an equilateral triangle with side length $s$ is

$$A = \frac{1}{2}sh = \frac{\sqrt{3}}{4}s^2$$

So if $s_{\mathrm{I}}$ is the side length of triangle I, then the area of triangle I is

$$A_{\mathrm{I}} = \frac{\sqrt{3}}{4}s_{\mathrm{I}}^2$$

Since the altitude of triangle I is the side of triangle II, the side length of triangle II is

$$s_{\mathrm{II}} = \frac{\sqrt{3}s_{\mathrm{I}}}{2}$$

so the area of triangle II is

$$A_{\mathrm{II}} = \frac{\sqrt{3}}{4}s_{\mathrm{II}}^2 = \left(\frac{\sqrt{3}}{4}\right)\left(\frac{\sqrt{3}s_{\mathrm{I}}}{2}\right)^2 = \frac{3}{4}\left(\frac{\sqrt{3}s_{\mathrm{I}}}{2}\right) = \frac{3}{4}A_{\mathrm{I}}$$

In the same way the area of triangle III is

$$A_{\mathrm{III}} = \frac{3}{4}A_{\mathrm{II}} = \frac{9}{16}A_{\mathrm{I}}$$

and finally

$$A_{\mathrm{IV}} = \frac{3}{4}A_{\mathrm{III}} = \frac{27}{64}A_{\mathrm{I}} = \frac{27}{32}$$

**Answer: The area of triangle IV is** $\frac{27}{32}$

2. If $s$ is the side length of the square and $c$ is the edge length of the cube, then $s^2 = 3$, so that $s = \sqrt{3} = 3^{1/2}$, and $c^3 = 5$, so that $c = \sqrt[3]{5}$. So the question is which is larger, $\sqrt{3}$ or $\sqrt[3]{5} = 5^{1/3}$. Raise each to the sixth power.

$$s^6 = \left(3^{1/2}\right)^6 = 3^3 = 27 \quad \text{and} \quad c^6 = \left(5^{1/3}\right)^6 = 5^2 = 25$$

Since $s^6 > c^6$ it must be true that $s > c$. So the side length of the square is greater than the edge length of the cube.

**Answer: The side length of square of area 3 is greater than the edge length of the cube of volume 5.**

---

3. Let $X = L \cdots DCBA6$ be the original number, and let $Y = 6L \cdots DCBA$ be the transformed number. We want to find the smallest number $X$ so that $Y = 4X$. Since $6 \times 4 = 24$, the leftmost digit in $Y$, $A$, must equal 4. Since $A = 4$, the number $X$ ends in 46 and $4 \times 46 = 184$. Thus, $B = 8$ and so $X$ ends in 846 with $4 \times 846 = 3384$. Thus, $C = 3$ and so $X$ ends in 3846 with $4 \times 3846 = 15384$. Thus, $D = 5$. At this point observe that $4 \times 153846 = 615384$. Hence, $X = 153846$. Note that we have found $L = 1$, which can be seen by noting that otherwise multiplying $X$ by 4 will make the leftmost digit greater than 6 or produce a carry.

**Alternate solution:** Let the number be $X = a_n a_{n-1} \cdots a_1 6$, where the $a_i$ are integers with $0 \leq a_i \leq 9$ but $a_n > 0$. Then, the number with the rightmost "6" moved to the leftmost position is $Y = 6a_n a_{n-1} \cdots a_1$. Hence

$$X = a_n \times 10^n + a_{n-1} \times 10^{n-1} + \cdots a_1 \times 10 + 6$$

and

$$Y = 6 \times 10^n + a_n \times 10^{n-1} + a_{n-1} \times 10^{n-2} + \cdots a_1$$

Since $Y = 4X$, we have

$$6 \times 10^n + a_n \times 10^{n-1} + a_{n-1} \times 10^{n-2} + \cdots a_1 = 4a_n \times 10^n + 4a_{n-1} \times 10^{n-1} + \cdots 4a_1 \times 10 + 24$$

Since $4a_i \times 10^i - a_i \times 10^{i-1} = a_i \left(40 \times 10^{i-1} - 10^{i-1}\right) = 39a_i \times 10^{i-1}$, rearranging this equation gives

$$39a_n \times 10^{n-1} + 39a_{n-2} \times 10^{n-1} + \cdots 39a_1 = 6 \times 10^n - 24$$

and dividing by 3 results in

$$13\left(a_n \times 10^{n-1} + a_{n-2} \times 10^{n-1} + \cdots a_1\right) = 2 \times 10^n - 8$$

So we need to find $n$ so that $2 \times 10^n - 8$ is divisible by 13, or equivalently, so that $2 \times 10^n$ has a remainder of 8 when divided by 13. Dividing the numbers 20, 200, 2000, 20000, and 200000 by 13 gives remainders of 7, 5, 11, 6, and 8. Thus, $200000 - 8 = 199992$ is divisible by 13. The division gives 15384. Hence

$$X = 153846 \quad \text{and} \quad Y = 615384$$

**Answer: The smallest such positive integer is 153846.**

4. The members of a committee sit at a circular table so that each committee member has two neighbors. Each member of the committee has a certain number of dollars in his or her wallet. The chairperson of the committee has one more dollar than the vice chairperson, who sits on his right and has one more dollar than the member on her right, who has one more dollar than the person on his right, and so on, until the member on the chair's left is reached. The chairperson now gives one dollar to the vice chair, who gives two dollars to the member on her right, who gives three dollars to the member on his right, and so on, until the member on the chair's left is reached. There are then two neighbors, one of whom has four times as much as the other.

   (a) Adrian, the person to the left of the chairperson, is the poorest person on the committee at first. Let $x$ be the number of dollars that Adrian has at first. Note that Alex, the person on Adrian's left has $x + 1$ dollars at first. Let $n$ be the number of members of the committee. After the money is redistributed, every person, except Adrian, has exactly one dollar less than they had before. Since there are $n-1$ members of the committee besides Adrian, as part of the redistribution of the money Adrian receives $n - 1$ dollars from Alex. So after the redistribution, Adrian has $x + n - 1$ dollars and Alex has $x$ dollars. Since Adrian and Alex must be the two neighbors described, we have

$$x + n - 1 = 4x \Rightarrow 3x = n - 1$$

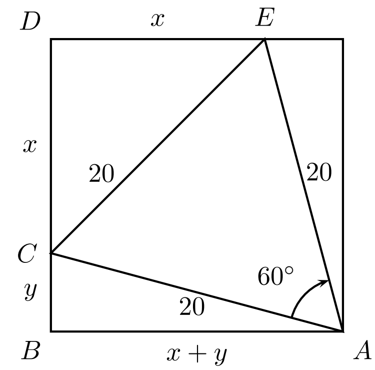
   So $n - 1$ must be a multiple of 3. The smallest value of $n$ that makes $n - 1$ a multiple of 3 is $n - 1 = 3 \Rightarrow n = 4$. Thus, the smallest possible number of members of the committee is 4 and, in this case, the poorest of the committee has one dollar at first.

   **Answer: The smallest possible number of members of the committee is 4 and the poorest member has 1 dollar at first.**

   (b) With at least 12 members on the committee, we need to find the smallest $n$ for which $n - 1$ is a multiple of 3. This is when $n-1 = 12 \Rightarrow n = 13$. In this case, $x = 4$. So with at least 12 members on the committee the smallest possible number of members is 13 and the poorest member has 4 dollars at first.

   **Answer: With at least 12 members on the committee the smallest possible number of members of the committee is 13 and the poorest member has 4 dollar at first.**

5. Let $x$ equal the length of the segments $\overline{CD}$ and $\overline{DE}$ in the diagram, let $y$ be the length of the segment $\overline{BC}$, and let $s$ be the side length of the equilateral triangle $ACE$. Then $x + y$ is the length of the side $AB$ and Pythagoras' Theorem gives




$$s^2 = 2x^2 \quad \text{and} \quad s^2 = (x+y)^2 + y^2$$

Since $s = 20$, this gives

$$x^2 = 200 \Rightarrow x = \sqrt{200} = 10\sqrt{2}$$

and

$$2x^2 = x^2 + 2xy + 2y^2 \Rightarrow 2y^2 + 20\sqrt{2}y - 200 = 0 \Rightarrow y^2 + 10\sqrt{2} - 100 = 0$$

Solving for $y$ gives

$$y = \frac{-10\sqrt{2} \pm \sqrt{200+400}}{2} = 5\left(-\sqrt{2} \pm \sqrt{6}\right)$$

Taking positive square root gives a positive value for $y$. Hence

$$y = 5\left(\sqrt{6} - \sqrt{2}\right)$$

and the side length of the square is

$$x + y = 5\sqrt{6} - 5\sqrt{2} + 10\sqrt{2} = 5\left(\sqrt{6} + \sqrt{2}\right)$$

**Alternate solution:** Using trigonometry, noting that $\angle BAC = 15°$, we have

$$x + y = 20\cos 15°$$

Using the half-angle formula

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$$

gives

$$\cos 15° = \sqrt{\frac{1+\cos 30°}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{3}}$$

so that

$$x + y = 10\sqrt{2+\sqrt{3}}$$

Note that the two answers are the same since

$$\left(\sqrt{6}+\sqrt{2}\right)^2 = 8 + 4\sqrt{3} = 4\left(2+\sqrt{3}\right) \quad \text{and} \quad \left[2\sqrt{2+\sqrt{3}}\right]^2 = 4\left(2+\sqrt{3}\right)$$

**Answer: The side length of the square is $5\left(\sqrt{6}+\sqrt{2}\right)$ or $10\sqrt{2+\sqrt{3}}$, which are equal.**

# Senior Final, Part A

1. There are $12 \times 12 = 144$ stamps of which $11 \times 4 = 44$ are on the edge. Thus, the probability of picking an edge stamp is

$$\frac{44}{144} = \frac{11}{36}$$

**Answer is (C).**

---

2. Five of the eight integers can be selected in

$$\binom{8}{5} = \frac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1} = 56$$

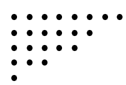
ways. The following 11 selections have sums less than 20:

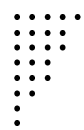
$$\{1,2,3,4,5\}, \{1,2,3,4,6\}, \{1,2,3,5,6\}, \{1,2,4,5,6\}, \{1,3,4,5,6\}, \{1,2,3,4,7\},$$

$$\{1,2,3,5,7\}, \{1,2,4,5,7\}, \{1,2,3,6,7\}, \{1,2,3,4,8\}, \{1,2,3,5,8\}$$

Thus there are $56 - 11 = 45$ have sums at least 20.

**Alternate solution:** A selection of five integers from the set $\{1,2,3,4,5,6,7,8\}$, such as $\{1,3,5,6,8\}$, can be visualized in a dot diagram as shown on the left below.





Reflecting the diagram along a diagonal gives the diagram on the right. This shows that the selection of exactly five distinct integers from the set $\{1,2,3,4,5,6,7,8\}$ is equivalent to selecting from 5 to 8 integers from the set $\{1,2,3,4,5\}$ with repetition allowed with each of the integers in the set selected at least once. Finally, since each of the numbers in $\{1,2,3,4,5\}$ must be selected at least once, we can select up to 3 integers from $\{1,2,3,4,5\}$ and then add one each of the $\{1,2,3,4,5\}$. Note that this includes the possibility of selecting none of the integers.

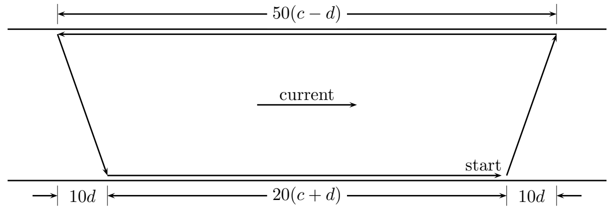
Since $1+2+3+4+5 = 15$, the problem above can be restated as: Find the number of ways of selecting from 0 to 3 numbers from $\{1,2,3,4,5\}$, with repetition allowed, so that the sum of the numbers is at least $20 - 15 = 5$. Or, as in the solution above, find the number of selections in which the sum is 4 or less, and subtract from 56. The selections are:

$$\begin{array}{ccccc} \{4\}, & \{3,1\}, & \{2,2\}, & \{2,1,1\}, & \{3\}, \\ \{2,1\}, & \{1,1,1\}, & \{2\}, & \{1,1\}, & \{1\} \end{array}$$

Note that the selection $\{4\}$ is equivalent to a selection in the original problem of $\{1,3,4,5,6\}$, with a sum of 19. Adding the selection with no elements, the empty set, gives 11 selections with a sum of 4 or less. This gives the same answer as the original solution.

**Answer is (B).**

3. Let $c$ be the speed of the canoe in still water and $d$ be the drift speed of the water. As the canoeist paddles across the river she drifts a distance of $10d$. Then her speed is $c-d$ as she paddles upstream for 50 minutes going a distance of $50(c-d)$. She then drifts $10d$ recrossing the stream. Then her speed is $c+d$ as she paddles with the current for 20 minutes going $20(c+d)$. See the diagram for the canoeist's path. If we make the direction of the current positive, then the total distance is

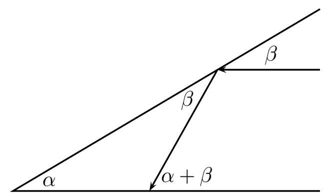



$$10d - 50(c-d) + 10d + 20(c+d) = 0 \Rightarrow 90d = 30c \Rightarrow c : d = 3 : 1$$

**Answer is (A).**

---

4. Let the angle between the mirrors be $\alpha$ and the angle which the incoming ray makes with the mirror be $\beta$. We see from the diagram at the right that the ray will make an angle of $\alpha+\beta$ with the next mirror. We see that the ray can make a maximum of 8 reflections by (for example) coming in with an angle of 10°, making 8 reflections and leaving at an angle of 170°.




**Answer is (D).**

---

5. If we place the origin at $C$ and the positive $x$-axis along $\overline{CB}$ then $B$ is at $(17,0)$, $D$ is at $(0,6)$, and $A$ is at $(-4,0)$. The centre of the circle is at the intersection of the perpendicular bisectors of $\overline{AD}$ and $\overline{BD}$. If we have points $(a,0)$ and $(0,b)$ then the midpoint is at $(a/2,b/2)$ and the slope of the perpendicular is $a/b$ and the equation of the perpendicular bisector is

$$\frac{y-b/2}{x-a/2} = \frac{a}{b} \Rightarrow y = \frac{a}{b}x + \frac{b^2-a^2}{2b}$$

We want to solve

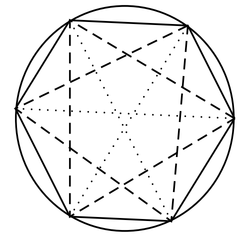
$$\begin{aligned} y &= \frac{17}{6}x + \frac{6^2-17^2}{12} \\ y &= \frac{-4}{6}x + \frac{6^2-4^2}{12} \\ \frac{17+4}{6}x &= \frac{17^2-4^2}{12} \\ x = \frac{13}{2} & \qquad y = -\frac{8}{3} \end{aligned}$$

Hence, the radius of the circle is

$$R = \sqrt{\left(\frac{13}{2}-4\right)^2 + \left(-\frac{8}{3}\right)^2} = \frac{65}{6}$$

**Answer is (A).**

6. If we first draw lines between adjacent points we get 6 regions plus the central one. If we then draw lines between alternate points we get a Star of David which divides the central region into 12 regions plus a new, smaller central region giving a total of 18 regions plus the central region. Drawing lines between opposite points divides each of the points of the star in two for a total of 6 new regions and the central region into 7 for a total of 31 regions.



**Alternate solution:** Note that if $\ell$ lines are drawn through a circle and the intersect in $s$ points, then the lines form $\ell + s + 1$ regions. You can see this by noting that one line divides the circle into 2 regions and adding one more line without an intersection point adds one more region. Adding an intersection point as well adds one more region. Now, each pair of points on the circle determines a line, giving a total of

$$\binom{6}{2} = 15$$

lines. Further, if you choose 4 points on the circle, and draw the 6 lines determined by them, you will see that only one pair of lines intersect. Thus, every set of 4 points on the circle determines an intersection point. Hence, the maximum possible number of distinct intersection points is

$$\binom{6}{4} = 15$$

Hence, the maximum possible number of regions is $15 + 15 + 1 = 31$. Note that the formula $\ell + s + 1$ for the number of regions produced by $\ell$ lines and $s$ intersection points could also be used to solve Senior Preliminary Problem 7.

**Answer is (D).**

---

7. Let $n_i$ be the number of artists who were nominated in at least $i$ categories. There were 103 categories with 5 nominations in each category, for a total of 515 nominations. We see that

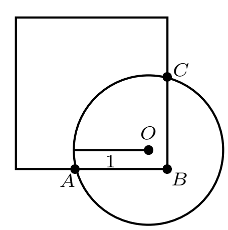
$$n_1 + n_2 + n_3 + n_4 + n_5 = 515$$

$$n_1 = 515 - 50 - 35 - 24 - 12 = 394$$

**Answer is (E).**

---

8. Place the origin at $B$ and the positive $x$-axis along $\overline{BA}$. Then $A$ is at $(a, 0)$ and $O$ is at $(c, c)$ where $c = \frac{1}{2\sqrt{2}}$. The radius of the circle is $r = 1$. Then




$$c^2 + (a - c)^2 = r^2 \Rightarrow a^2 - 2ac + 2c^2 - r^2 = 0$$

So that

$$a = \frac{2c \pm \sqrt{4c^2 - 4(2c^2 - r^2)}}{2} = c \pm \sqrt{r^2 - c^2}$$
$$= \frac{1}{2\sqrt{2}} \pm \sqrt{1 - \frac{1}{8}} = \frac{1 \pm \sqrt{7}}{2\sqrt{2}}$$

We obviously need to take the positive sign,

$$a = \frac{1 + \sqrt{7}}{2\sqrt{2}}$$

**Answer is (B).**

9. Let $H$ be the number of degrees the hour is from 12 on the clock at the time of interest, and let $M$ be the number of degrees for the minute hand. Further, let $t$ be the number of minutes after 8 o'clock when the minute hand and the hour hand are an equal angular distance on either side of the six. Then, since the hour hand moves at one half degree per minute and the minute hand moves at six degrees per minute, we have

$$H = 240 + \tfrac{1}{2}t$$
$$M = 6t$$

Now, since six on the clock is 180 degrees from the 12, for the minute hand and the hour hand to an equal angular distance on either side of the six, we must have

$$H - 180 = 180 - M \Rightarrow H + M = 360 \Rightarrow 240 + \tfrac{1}{2}t + 6t = 360 \Rightarrow \tfrac{13}{2}t = 120$$

Solving for $t$ gives

$$t = \frac{240}{13} = 18 + \frac{6}{13}$$

Converting $\frac{6}{13}$ minutes to seconds gives

$$\frac{360}{13} = 27 + \frac{9}{13}$$

Since $\frac{9}{13} > \frac{1}{2}$, rounding the nearest second gives 28 seconds.

**Answer is (B).**

---

10. The probability of any given two ball draw is

$$p = \frac{1}{6} \times \frac{1}{5} = \frac{1}{30}.$$

However, since we are not concerned about the order in which the ball are drawn (i.e. the event $(2, 7)$ is equivalent to $(7, 2)$), the probability of drawing any given pair of balls is $p_2 = 2p = \dfrac{1}{15}$. If we consider the sum of the numbers on any pair of balls, then the 15 possible outcomes are summarised in the table below.

| | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| 2 | - | **5** | 6 | **7** | 8 | 9 |
| 3 | - | - | **7** | 8 | 9 | 10 |
| 4 | - | - | - | 9 | 10 | **11** |
| 5 | - | - | - | - | **11** | 12 |
| 6 | - | - | - | - | - | **13** |
| 7 | - | - | - | - | - | - |

Since six of these numbers are prime, the probability that the sum of the numbers on the selected balls is a prime number is $\dfrac{6}{p_2} = \dfrac{2}{5}$

**Answer is (C).**

# Senior Final, Part B

1. Let $k$ be the first integer in the sequence and $n$ be the number of integers in the sequence. Then, the sum of the consecutive integers is

$$k + (k+1) + (k+2) + \cdots + (k+n-1) = nk + 1 + 2 + \cdots + n - 1 = nk + \tfrac{1}{2}n(n-1) = 100$$

Thus, we must have

$$2nk + n(n-1) = n(2k+n-1) = 200 \Rightarrow 2k + n - 1 = \frac{200}{n}$$

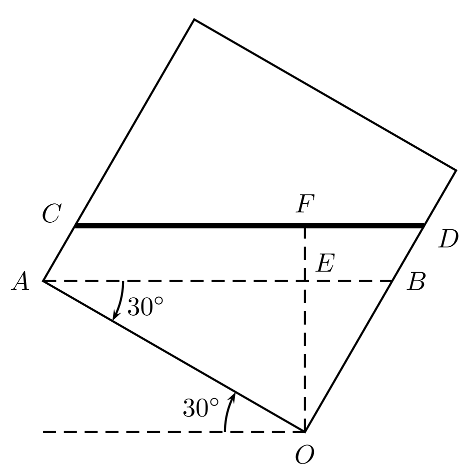
So, $n$ must be a divisor of 200 with $n > 1$: 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, or 200. Consider the following table

| $n$ | Equation for $k$ | $k$ |
|---|---|---|
| 2 | $2k+1=100$ | no integer solution |
| 4 | $2k+3=50$ | no integer solution |
| 5 | $2k+4=40$ | $k=18$ |
| 8 | $2k+7=25$ | $k=9$ |
| 10 | $2k+9=20$ | no integer solution |
| 20 | $2k+11=10$ | no integer solution |
| 25 | $2k+24=8$ | $k=-8$ |
| 40 | $2k+39=5$ | $k=-17$ |
| 50 | $2k+49=4$ | no integer solution |
| 100 | $2k+99=2$ | no integer solution |
| 200 | $2k+199=1$ | $k=-99$ |

Thus, there are five sequences of consecutive integers whose sum is 100. Note that the sequence for $n = 25$ is the sequence for $n = 8$ with the 17 consecutive integers from $-8$ to 8, which add up to zero, included. In the same way, the sequence for $n = 40$ is the sequence for $n = 5$ with the 35 consecutive integers from $-17$ to 17, which add up to zero, included.

**Answer: There are five sequences of consecutive integers whose sum is 100.**

2. The diagram shows a view of the tank looking at the face that is rotated. The line $CD$ is the surface of the water in the tank, and the length of the segment $OF$ is the depth of the water. Since the tank is half filled, the **volume** of the water in the tank is $\frac{1}{2}\,\text{m}^3$. Then, since the length of each edge of the tank is one meter, the **area** of the cross-section perpendicular to the edge along which the tank is rotated is $\frac{1}{2}\,\text{m}^2$. In particular, the area of the quadrilateral $OACD$ is $\frac{1}{2}$. The quadrilateral $OACD$ can be divided into triangle $OAB$ and parallelogram $ABDC$. Let $h_1 = \overline{OE}$ be the height of triangle $OAB$, and let $h_2 = \overline{EF}$ be the perpendicular distance between two parallel sides of the parallelogram $ABDC$. Then, $h_1 + h_2 = \overline{OF}$ is the depth of the water in the tank.



...Problem 2 continued

Now, since triangle $OAE$ is a 30-60-90 triangle with hypotenuse $OA$ equal to one, we have $h_1 = \frac{1}{2}$. Further, since $OAB$ is a 30-60-90 triangle with the side $OA$ adjacent to the 30° angle equal to one, we have the hypotenuse $\overline{AB} = \frac{2}{\sqrt{3}}$. Thus, the area of triangle $OAB$ is

$$A_{OAB} = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

Since the sum of the areas of triangle $AOB$ and parallelogram $ABDC$ is $\frac{1}{2}$, we have

$$A_{ABDC} = \frac{1}{2} - \frac{\sqrt{3}}{6} = \frac{3-\sqrt{3}}{6}$$

But

$$A_{ABDC} = h_2 \cdot \overline{AB} = h_2\left(\frac{2}{\sqrt{3}}\right)$$

So that

$$h_2\left(\frac{2}{\sqrt{3}}\right) = \frac{3-\sqrt{3}}{6} \Rightarrow h_2 = \frac{3\sqrt{3}-3}{12} = \frac{\sqrt{3}-1}{4}$$

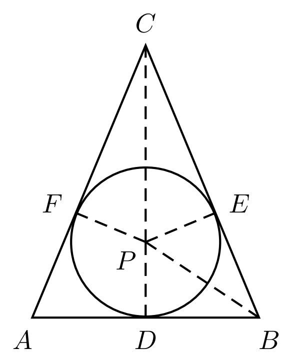
and the depth of water in the tank is

$$h_1 + h_2 = \frac{1}{2} + \frac{\sqrt{3}-1}{4} = \frac{2+\sqrt{3}-1}{4} = \frac{\sqrt{3}+1}{4}$$

**Answer: The depth of the water in the tank is** $\frac{\sqrt{3}+1}{4}$

---

3. Consider the diagram below




In the diagram $P$ is the centre of the inscribed circle. Let $r_i = \overline{PD} = \overline{PE} = \overline{PF}$ be the radius of the inscribed circle. In the isosceles triangle $ABC$ the line $CD$ is the perpendicular bisector of the side $AB$. Thus, triangle $BCD$ is a right triangle and by Pythagoras' theorem

$$\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2 \Rightarrow 13^2 = 5^2 + \overline{CD}^2 \Rightarrow \overline{CD}^2 = 169 - 25 = 144$$

Thus, $\overline{CD}^2 = 12$. Further, triangles $OBD$ and $OBE$ are both right triangles with two equal sides: $PB$ which is common to both triangles, and sides $PD$ and $PE$ which are both radii of the circle. So the two triangles are congruent with $\overline{BD} = \overline{BE} = 5$, and, thus $\overline{CE} = 13 - \overline{BE} = 13 - 5 = 8$. Since line $BC$ is tangent to the circle, it is perpendicular to the radius $PE$, so that triangle $PCE$ is a right triangle with sides $\overline{PE} = r_i$, $\overline{CE} = 8$, and $\overline{PE} = 12 - r_i$. Using Pythagoras' theorem again gives

$$r_i^2 + 8^2 = (12 - r_i)^2 = 144 - 24r_i + r_i^2 \Rightarrow 24r_i = 144 - 64 = 80 \Rightarrow r_i = \frac{80}{24} = \frac{10}{3}$$

**Answer: The radius of the inscribed circle is** $\frac{10}{3}$**.**

4. First note that $155 = 31 \cdot 5$ and $203 = 29 \cdot 7$, where 5, 7, 29, and 31 are all primes. Since $a$, $b$, $c$, $d$, and $e$ are integers, $b$ must be a divisor of 155 and $c$ must be a divisor of 203. Thus there are four cases to consider: $b = 5$ and $c = 7$, $b = 5$ and $c = 29$, $b = 31$ and $c = 7$, or $b = 31$ and $c = 29$.

For $b = 5$ and $c = 7$ the second and third equations give

$$a + 7 + d + e = 31 \Rightarrow a + d + e = 24 \quad \text{and} \quad a + 5 + d + e = 29 \Rightarrow a + d + e = 24$$

In either case $d + e = 24 - a$. Then the first equation gives

$$a\,(5 + 7 + d + e) = 128 \Rightarrow a\,(12 + 24 - a) = 128 \Rightarrow a^2 - 36a + 128 = (a - 32)(a - 4) = 0$$

So that $a = 32$ or $a = 4$. If $a = 32$, then $d + e = -8$ which is impossible since $d, e > 0$. For $a = 4$, $d + e = 20 \Rightarrow e = 20 - d$ and the fourth equation gives

$$d(4 + 7 + 5 + e) = d(16 + 20 - d) = d(36 - d) = 243 \Rightarrow d^2 - 36d + 243 = (d - 27)(d - 9) = 0$$

So that $d = 27$ or $d = 9$. If $d = 27$, then $e = -7$. So the only possibility is $d = 9$ and then $e = 11$. Then we must have $a = 4$, $b = 5$, $c = 7$, $d = 9$, and $e = 11$.

Checking the last equation gives

$$11\,(4 + 5 + 7 + 11) = 11 \cdot 25 = 275$$

For $b = 5$ and $c = 29$ the second and third equations give

$$a + 29 + d + e = 31 \Rightarrow a + d + e = 2 \quad \text{and} \quad a + 5 + d + e = 7 \Rightarrow a + d + e = 2$$

Reject this case, since three positive integers cannot add up to 2. For $b = 31$ and $c = 7$ or $b = 31$ and $c = 29$ the second and third equations give

$$a + 7 + d + e = 5 \Rightarrow a + d + e = -2 \quad \text{and} \quad a + 31 + d + e = 29 \Rightarrow a + d + e = -2$$

or

$$a + 7 + d + e = 5 \Rightarrow a + d + e = -24 \quad \text{and} \quad a + 31 + d + e = 29 \Rightarrow a + d + e = -24$$

Reject either case since all of the integers are positive.

Note that for $b = 1$ or $b = 155$, or $c = 1$ or $c = 203$ the second and third equations are inconsistent.

**Alternate solution:** First note that since $a$, $b$, $c$, $d$, and $e$ are integers greater than one, i.e., $a, b, c, d, e \geq 2$, the sum of any four of them is at least equal to 8. Then, since $b(a+c+d+e) = 155 = 5\times 31$, where 5 and 31 are prime, we must have $b = 5$ and $a + c + d + e = 31$, since $a + c + d + e \geq 8$. In the same way, $c(a + b + d + e) = 203$ gives $c = 7$ and $a + b + d + e = 29$. Either of these results gives $a + d + e = 24 \Rightarrow a + b + c + d + e = 36$. Then the equation $a(b + c + d + e) = 128$ becomes

$$a(36 - a) = 128 \Rightarrow a^2 - 36a + 128 = (a - 32)(a - 4) = 0$$

The only possible solution is $a = 4$, since $a = 32$ makes $a + b + c + d + e \geq 40$. Therefore, $a + b + c = 16$ and the equation $e(a + b + c + d) = 275$ becomes $e(16 + d) = 275$ where $d + e = 36 - a - b - c = 20$. Since $275 = 11 \times 25$ and $16 + d \geq 18$, we must have $e = 11$ and $d = 25 - 16 = 9$. Note that the other factoring of $275 = 5 \times 55$, makes $d = 39$ which is too large.

**Answer: The five numbers are $a = 4$, $b = 5$, $c = 7$, $d = 9$, and $e = 11$.**

---

5. Since $2^9 = 512$, $2^{10} = 1024$ and $2^{11} = 2048$ we see that 2006 is in the tenth row. The $y$-axis comes between 1024+511=1535 and 1536. Thus $x$-coordinate of 2006 is $1 + 2 \times (2006 - 1536) = 941$. Thus 2006 is at $(941, 20)$.

**Answer: 2006 is at $(941, 20)$.**