BRITISH COLUMBIA COLLEGES
High School Mathematics Contest
2003 Solutions

Junior Preliminary

1. Solve the equation

$$\frac{1}{1 - \frac{1}{x}} = -2$$

Simplify the complex fraction by multiplying numerator and denominator by x.

$$\frac{x}{x-1} = -2 \quad \Longrightarrow \quad x = -2(x-2) \quad \Longrightarrow \quad 3x - 2 = 0$$

The only solution to this equation is $x = \frac{2}{3}$.

<u>**answer is (c)**</u>

2. Let A_1, ℓ_1 and w_1 be the area, length and width of the original rectangle. Then $A_1 = \ell_1 w_1$. For the new rectangle

$$\ell_2 = (1 + 0.25)\ell_1 \quad \text{and} \quad w_2 = (1 - 0.25)w_1$$

Then the area of the new rectangle is

$$A_2 = \ell_2 w_2 = (1 + 0.25)\ell_1 \cdot (1 - 0.25)w_1 = \left(1 - 0.25^2\right)\ell_1 w_1 = (1 - 0.0625)A_1$$

Thus, there is a 6.25% decrease in the area.

<u>**answer is (c)**</u>

3. Rewriting the two given inequalities gives

$$x - y > x \quad \Longrightarrow \quad x > x + y \quad \Longrightarrow \quad 0 > y \quad \Longrightarrow \quad y < 0$$

and

$$x + y > x \quad \Longrightarrow \quad x > 0$$

<u>**answer is (e)**</u>

4. We only need to find the number of multiples of $15 \cdot 16 = 240$ between 1400 and 2400, inclusive. But

$$\frac{1400}{240} = 5\tfrac{5}{6} \quad \text{and} \quad \frac{2400}{240} = 10$$

So the only multiples of 240 that are between 1400 and 2400 are those with 6, 7, 8, 9, and 10 as multipliers.

<u>**answer is (e)**</u>

5. If $2^6 + m^n = 2^7 = 2 \cdot 2^6$, then $m^n = 2^6$. Further,

$$2^6 = \left(2^2\right)^3 = \left(2^3\right)^2 = \left(2^6\right)^1$$

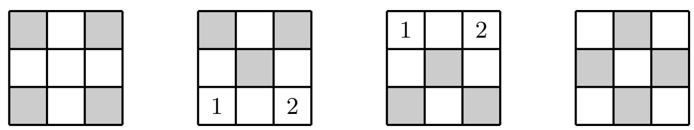
Thus, the possible values for m and n are

$$\begin{aligned} m = 2,\ n = 6 &\implies x = 8 \\ m = 2^2 = 4,\ n = 3 &\implies x = 7 \\ m = 2^3 = 8,\ n = 2 &\implies x = 10 \\ m = 2^6 = 64,\ n = 1 &\implies x = 65 \end{aligned}$$

Thus, the sum of the possible values for x is $8 + 7 + 10 + 65 = 90$.

answer is (a)

6. The possible shadings of the grid with none of four shaded squares having an edge in common are



The two grids in the centre give two possible shadings, as indicated by the numbering. Thus, there is a total of 6 possible shadings.

answer is (d)

7. Since the diagonal of one of the small squares is 2, the length of each of the sides must be $\sqrt{2}$. Hence, the length of the rectangle is $3\sqrt{2}$ and the width is $2\sqrt{2}$. Therefore, the length of the diagonal of the rectangle is

$$\sqrt{\left(3\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2} = \sqrt{9 \cdot 2 + 4 \cdot 2} = \sqrt{26}$$

answer is (b)

8. The remainders when increasing powers of 3 are divided by 5 are

3^0 remainder on division by 5 is 1
3^1 remainder on division by 5 is 3
3^2 remainder on division by 5 is 4
3^3 remainder on division by 5 is 2
3^4 remainder on division by 5 is 1

The pattern repeats in groups of 4. Since 22 has a remainder of 2 when divided by 4, the remainder when 3^{22} is divided by 5 is that same as that when 3^2 is divided by 5. Thus, 3^{22} has a remainder of 4 when divided by 5.

answer is (a)

9. The number of squares in Figure n is $2n + 1$. Thus, the number of squares in the 2003^{rd} figure is

$$2(2003) + 1 = 4007$$

answer is (e)

10. The triangles with perimeter 13 having integer side lengths are: 1-6-6, 2-5-6, 3-4-6, 3-5-5, and 4-4-5. Only two of these are scalene triangles: 2-5-6 and 3-4-6. For perimeter 12 the triangles are: 2-5-5, 3-4-5, and 4-4-4. Only one is scalene: 3-4-5. For perimeter 11 the only triangle is: 2-4-5. This is scalene. For perimeter 10 the triangles are: 2-4-4 and 3-3-4. Neither is scalene. For perimeter 9 the triangles are: 1-4-4, 2-3-4, and 3-3-3. Only one is scalene; 2-3-4. For perimeters less than 9 the triangles are: 2-3-3, 1-3-3, 2-2-3, 2-2-2, 1-2-2, and 1-1-1. None of these is scalene. Thus, there are 5 scalene triangles with perimeter at most 13 that have integer side lengths.

    Alternate solution: Let a, b, and c be the sides of the triangle. Since the triangle is scalene, we may assume that $a < b < c$. Now, in any triangle we must have $c < a + b$. Since a, b, and c are all integers, this inequality prohibits $a = 1$, since $b < c$ means that $c \geq b + 1$ and $a = 1$ means that $c < b + 1$, which is a contradiction. Furthermore, for $a = 2$, we must have $c = b + 1$, and for $a = 3$, we must have $c = b + 1$ or $c = b + 2$. Finally, since $a > 3$ implies that $a + b + c$ is at least $4 + 5 + 6 = 15$, we need not consider $a > 3$. Thus we have,

$a < b < c$	perimeter
2 - 3 - 4	9
2 - 4 - 5	11
2 - 5 - 6	13
3 - 4 - 5	12
3 - 4 - 6	13

    for a total of 5 scalene triangles with perimeter of at most 13.

    answer is (a)

11. Since the digit sum of 2003 is 5, we are looking for four-digit numbers less than 2003 with a digit sum less than 5, i.e., equal to 4 or 3 or 2 or 1. The four-digit numbers less than 2003 with a digit sum of 4 are: 1003, 1012, 1021, 1030, 1102, 1111, 1120, 1201, 1210, 1300, and 2002. There are 11 such numbers. The four-digit numbers less than 2003 with a digit sum of 3 are: 1002, 1011, 1020, 1101, 1110, 1200, and 2001. There are 7 such numbers. The four-digit numbers less than 2003 with a digit sum of 2 are: 1001, 1010, 1100, and 2000. There are 4 such numbers. The four-digit number less than 2003 with a digit sum of 1 is: 1000. Altogether, there are $11 + 7 + 4 + 1 = 23$ four-digit numbers less than 2003 with a digit sum less than that of 2003.

    answer is (c)

12. The possible outcomes for a pair of dice can be enumerated as ordered pairs of numbers (x, y) where $x = 1 \ldots 6$ and $y = 1 \ldots 6$. There are 36 such ordered pairs, each of which is equally likely. The ordered pairs that give a total of 6 on the two dice are: $(1, 5)$, $(2, 4)$, $(3, 3)$, $(4, 2)$, and $(5, 1)$. There are 5 ordered pairs, so the probability of getting a total of 6 is $\frac{5}{36}$.

    answer is (d)

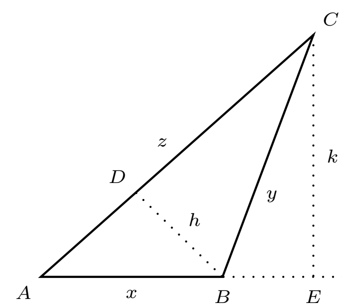
13. We must consider three separate cases:

$$\begin{aligned} x < -3 &\implies -(x+3) - (x-5) = 12 \implies -2x = 10 \implies x = -5 \\ -3 \leq x < 5 &\implies (x+3) - (x-5) = 12 \implies 8 = 12 \quad \text{no solution} \\ x \geq 5 &\implies (x+3) + (x-5) = 12 \implies 2x = 14 \implies x = 7 \end{aligned}$$

    Thus, there are two solutions to the equation $|x + 3| + |x - 5| = 12$, both of which are integers.

    answer is (d)

14. In the triangle ABC shown side $\overline{AB} = x$, $\overline{BC} = y$, and $\overline{AC} = z$. The shortest altitude is the perpendicular distance between vertex B and side AC. This is shown as $\overline{BD} = h$ in the diagram. The longest altitude is the perpendicular distance between vertex C and the (extension of) side AB. This is shown as $\overline{CE} = k$ in the diagram. Since $\angle CAE$ is common to both triangles ACE and ADB and both are right triangles, they are similar. Thus,



$$\frac{k}{z} = \frac{h}{x} \quad \Longrightarrow \quad k = \frac{zh}{x}$$

Alternate solution: Since the area of triangle ABC is one-half the product of any side of the triangle and the altitude to that side, we see that the longest side must be paired with the shortest altitude, and vice versa. Therefore, the area is

$$\tfrac{1}{2}zh = \tfrac{1}{2}xk$$

where k is the length of the longest altitude. Solving for k we get

$$k = \frac{zh}{x}.$$

answer is (d)

15. Each trailing zero in the product is produced by a 2 and a 5. Since twos appear more often than fives (in proceeding from the smallest factor to the largest), we see that the number of trailing zeroes is determined by the number of fives that appear as factors in the product. These only appear at the multiples of 5. The multiples of 5 to be considered are: 45, 40, 35, 30, 25, 20, 15, 10, and 5. Each of these, except for 25 contributes one trailing zero, while 25 contributes 2. Thus, the number of trailing zeros is 10.

answer is (c)

Senior Preliminary

1. Let x be the price charged and y be the quantity sold. Then the income is xy. If the price increase by the fraction p, the new price is $x + px = x(1 + p)$. If the quantity sold is decreased by the fraction d, then the new quantity sold is $y - dy = y(1 - d)$. If the income is to be the same, we must have

$$[x(1+p)]\,[y(1-d)] = xy \quad \Longrightarrow \quad xy(1+p)(1-d) = xy \quad \Longrightarrow \quad (1+p)(1-d) = 1.$$

Solving for d gives

$$1 - d = \frac{1}{1+p} \quad \Longrightarrow \quad d = 1 - \frac{1}{1+p} = \frac{1+p-1}{1+p} = \frac{p}{1+p}.$$

answer is (c)

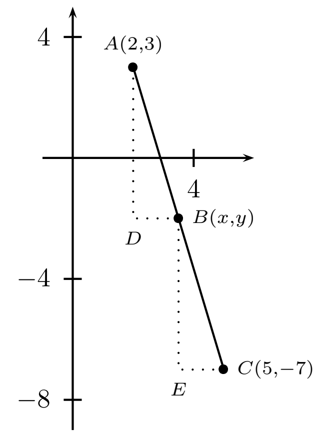
2. In the diagram, triangles ABD and BCE are similar triangles. Thus

$$\frac{x-2}{5-x} = \frac{\overline{AB}}{\overline{BC}} = \frac{3}{7} \implies 7x - 14 = 15 - 3x \implies x = \frac{29}{10},$$

and

$$\frac{3-y}{y-(-7)} = \frac{\overline{AB}}{\overline{BC}} = \frac{3}{7} \implies 21 - 7y = 3y + 21 \implies y = 0.$$

Thus, $x + y = \frac{29}{10}$.



answer is (b)

3. Obviously, objects with weights 1, 3, 4, 9, 10, 12, and 13 kilograms can be weighed, since this just involves putting the object being weighed on one pan and one or more of the given weights on the other pan. Further, no object weighing more than 13 kilograms can be weighed. Objects with other weights can be weighed as follows (let X be the object being weighed):

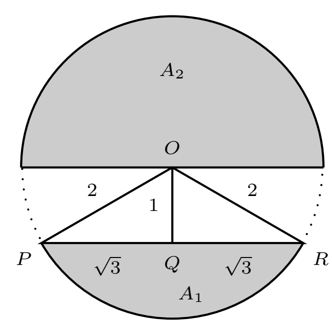
2 kg :	1 kg plus X in one pan	– 3 kg in the other
5 kg :	1 kg plus 3 kg plus X in one pan	– 9 kg in the other
6 kg :	3 kg plus X in one pan	– 9 kg in the other
7 kg :	3 kg plus X in one pan	– 1 kg plus 9 kg in the other
8 kg :	1 kg plus X in one pan	– 9 kg in the other
11 kg:	1 kg plus X in one pan	– 3 kg plus 9 kg in the other

Thus, an object with any weight between 1 and 13 kg can be weighed.

answer is (d)

4. We want the sum of areas A_1 and A_2 in the diagram. Area A_2 is the area of a semi-circle of radius $r = 2\,m$. So $A_2 = \frac{1}{2}\pi \cdot 2^2 = 2\pi$. Since the radius of the circle is 2 m and the width of the path is 1 m, the two triangles OPQ and OQR are both 30°-60°-90° triangles. Thus, the central angle, $\angle POR$, of circular sector OPR is $\frac{2\pi}{3}$ and area A_1 is the area of the circular sector less the area of the triangle OPR. Hence,

$$A_1 = \frac{1}{2}\left(2^2\right)\left(\frac{2\pi}{3}\right) - \frac{1}{2}\left(2\sqrt{3}\right) = \frac{4\pi}{3} - \sqrt{3}.$$



So the area remaining is

$$A_1 + A_2 = \frac{4\pi}{3} - \sqrt{3} + 2\pi = \frac{10\pi}{3} - \sqrt{3}.$$

answer is (a)

5. Imagine that the large circle has been cut open at point B and laid out flat as a straight line of length 18π (the circumference of the big circle); let us put several copies end-to-end. Now let x be the number of rotations that the small circle makes between successive times when A and B are in contact, and let y be the number of lengths of 18π units before A and B are again in contact. Since the circumference of the small circle is 4π, we see that

$$4\pi x = 18\pi y \quad \Longrightarrow \quad 2x = 9y.$$

When A and B come into contact x and y are integers. The smallest integers x and y that satisfy this equation are $x = 9$ and $y = 2$. Thus, the small circle makes 2 trips around the big circle before coming in contact with B again, and since the radius of the circle generated by the centre of the small circle is $9 + 2 = 11$, the total distance travelled by the centre of the small circle is $(22\pi)(2) = 44\pi$.

answer is (c)

6. Let a_1, a_2, ..., a_P be the distinct prime factors of N. Then a positive integer divisor of N is of the form

$$a_1^{n_1} \cdot a_2^{n_2} \cdots a_P^{n_P},$$

where n_i can only take the values 0 or 1, for $i = 1, \ldots, P$. Thus, there are 2^P positive integer divisors of N.

answer is (e)

7. Let n be the number of people who voted and a the number who voted for Anthony up to the time when he had 45% of the vote. Then

$$\frac{a}{n} = \frac{45}{100} = \frac{9}{20} \quad \Longrightarrow \quad 20a = 9n.$$

From the above equation, we note that n must be even. Let x be the number of people who voted for during the next five minutes, and let b be the number who voted for Anthony during that five-minute period. Then we have

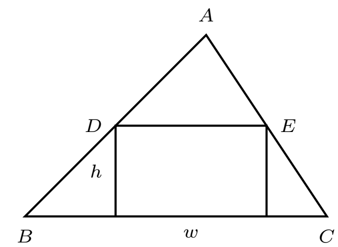
$$\begin{aligned} \frac{a+b}{n+x} = \frac{3}{10} \quad &\Longrightarrow \quad 10a + 10b = 3n + 3x \\ &\Longrightarrow \quad 20a + 20b = 9n + 20b = 6n + 6x \\ &\Longrightarrow \quad 3n + 20b = 6x. \end{aligned}$$

We note that x is as small as possible when b is as small as possible. Clearly, the equation above has a solution if $b = 0$, which gives $n = 2x$. Thus, $20a = 18x$, or $10a = 9x$. The smallest integer solution to this equation is $x = 10$ and $a = 9$.

Note: Because of the particular percentages chosen here, one can also arrive at this conclusion by making the assumption early on that x will be as small as possible if no one voted for Anthony during that five-minute period, but this assumption might NOT work if we had different percentages given in the problem. Why?

answer is (c)

8. Let h be the height and w be the width of the inscribed rectangle. The triangles ABC and ADE are similar, so that



$$\frac{w}{12-h} = \frac{20}{12} = \tfrac{5}{3} \quad\Longrightarrow\quad w = \tfrac{5}{3}(12-h).$$

Thus, the area of the rectangle is

$$A = wh = \tfrac{5}{3}h(12-h)$$

To maximize the expression $\frac{5}{3}h(12-h)$, it is sufficient to maximize the expression $h(12-h) = 12h - h^2$, which we can do by completing the square:

$$12h - h^2 = -\left(h^2 - 12h + 36 - 36\right) = -\left(h-6\right)^2 + 36.$$

This expression is clearly maximized when $x = 6$, and maximum value is 36. Therefore, the maximum area is $A = \frac{5}{3} \cdot 36 = 60$.

answer is (a)

9. The total number of equally likely outcomes when tossing a fair die three times is $6^3 = 216$. The outcomes, listed as ordered triples of numbers, in which the three numbers tossed are in strictly descending order are: (6,5,4), (6,5,3), (6,5,2), (6,5,1), (6,4,3), (6,4,2), (6,4,1), (6,3,2), (6,3,1), (6,2,1), (5,4,3), (5,4,2), (5,4,1), (5,3,2), (5,3,1), (5,2,1), (4,3,2), (4,3,1), (4,2,1), and (3,2,1). There 20 such outcomes. So the probability is $\frac{20}{6^3} = \frac{4}{3\cdot 3\cdot 6} = \frac{5}{54}$.

answer is (b)

10. Measuring angles clockwise in degrees from 12 o'clock, we observe that at 3:00 the hour hand is at $h = 90$ and the minute hand is at $m = 0$. The hour hand travels at 360° in 12 hours, so it travels 30° in one hour and $\frac{1}{2}^\circ$ in one minute. The minute hand travels 360° in one hour and so 6° in one minute. Thus, t minutes after 3:00 we have

$$h = 90 + \tfrac{1}{2}t \quad\text{and}\quad m = 6t.$$

When the hour and minute hands are again perpendicular, $m - h = 90$. This gives

$$6t - 90 - \tfrac{1}{2}t = 90 \quad\Longrightarrow\quad \tfrac{11}{2}t = 180 \quad\Longrightarrow\quad t = \frac{360}{11}.$$

Since t is the number of minutes past 3:00, the number of minutes past 3:30 is

$$t - 30 = \frac{360}{11} - 30 = \frac{30}{11}.$$

answer is (c)

11. The number of small triangles in Figure n is twice the sum of the first n odd integers, which sum is n^2. Thus, the number of small triangles in Figure n is $2n^2$. So for the 2003$^{\text{rd}}$ figure, the number of small triangles is

$$2(2003)^2 = 2(6009 + 2000 \cdot 2003) = 2(6009 + 4006000) = 2(4012009) = 8024018.$$

Alternate solution: Remove all of the horizontal lines from the diagram. Then we end up with an arrangement of small diamonds (rhombuses) obtained by dividing the sides of the outer rhombus into n equal segments. Thus, in Figure n there are n^2 diamonds, each of which was created from 2 small triangles, for a total of $2n^2$ small triangles in Figure n. This gives the same answer as above.

answer is (d)

12. The number of handshakes if each father shakes with everyone except himself is the number of ways of choosing two people from the group of 24. This is

$$\binom{24}{2} = \frac{24 \cdot 23}{2} = 276.$$

Of these, 12 involve a handshake of a father with his daughter. Thus, the total number of handshakes is $276 - 12 = 264$. Of these, the number that do not involve any fathers is the number of handshakes just between the 12 daughters. The number of such handshakes is

$$\binom{12}{2} = \frac{12 \cdot 11}{2} = 66.$$

Therefore, the number of handshakes that involve at least one father is

$$264 - 66 = 198$$

answer is (b)

13. Consider two cases for shading the grid with three squares shaded but with no two shaded squares having an edge in common.

    i) One row of the grid has two shaded squares. There is only one way for one row to have two shaded squares, the two ends of the row are shaded. If the top or bottom row has two shaded square, there are four squares in the grid that can be third shaded square. This gives a total of 8 shadings. If the middle row has two shaded squares, there are two squares in the grid that can be the third shaded square. This gives two more shadings. This gives a total of $8 + 2 = 10$ shadings in which one row of the grid has two shaded squares.

    ii) No row has two shaded squares. In this case each row has exactly one shaded square. There are three squares that can be shaded in the top row. Once the shaded square in the top row is shaded, there are two squares in the middle row that can be shaded. Once the square in the middle row is shaded, there are two squares in the bottom row that can be shaded. This gives a total of $3 \cdot 2 \cdot 2 = 12$ shadings in which no row has two shaded squares.

The two cases above cover all possibilities and adding the two gives a total of $10 + 12 = 22$ possible shadings.

Alternate solution: Consider 5 cases:

i) The central square of the grid is shaded. The other 2 squares must be corner squares and we must choose 2 of the 4 available, for a total of $\binom{4}{2} = 6$ possibilities.

In the remaining cases we assume the central square is unshaded.

ii) Exactly one corner square is shaded. The other 2 shaded squares must be the middle squares on the 2 sides not adjacent to the shaded corner. There is one such arrangement for each corner selected, for a total of 4 possibilities.

iii) Exactly 2 corner squares are shaded. These two shaded squares must appear in the same row (or column), and the third shaded square must be the middle square in the row (or column) opposite the one containing the shaded corners. There is exactly one possibility for each of the 4 pairs of shaded corners, for a total of 4 possibilities.

iv) Exactly 3 corner squares are shaded. This configuration is determined by choosing the corner that is unshaded. Thus, again there are 4 possibilities.

v) No corner is shaded. We must choose 3 of the 4 middle squares along the 4 sides, which is the same as choosing which of them not to shade. Thus, again there are 4 possibilities.

This gives a total of $6 + 4 + 4 + 4 + 4 = 22$ possible shadings.

answer is (d)

14. There are several ways to find the value of the sum. One way is to combine terms into groups of three which gives

$$(1+3-5)+(7+9-11)+(13+15-17)+\cdots+(61+63-65) = \underbrace{-1+5+11+\cdots+59}_{11 \text{ terms}}$$
$$= -1+6-1+12-1+\cdots+60-1 = 6(1+2+3+\cdots+10)-11$$
$$= 6\left(\frac{10\cdot 11}{2}\right) - 11 = 6\cdot 55 - 11 = 330 - 11 = 319.$$

answer is (e)

15. If we simply count the number of multiples of 15 and the number of multiples of 16 between 1400 and 2400, inclusive, we will count the multiples of both twice. But the multiples of both 15 and 16 are multiples of $15\cdot 16 = 240$. Now

$$\frac{1400}{15} = 93\tfrac{1}{3}, \quad \frac{2400}{15} = 160, \quad \frac{1400}{16} = 87\tfrac{1}{2}, \quad \frac{2400}{16} = 150.$$

Thus, the number of multiples of 15 between 1400 and 2400 is $160-93=67$ and the number of multiples of 16 between 1400 and 2400 is $150-87=63$. Further,

$$\frac{1400}{240} = 5\tfrac{5}{6} \quad \text{and} \quad \frac{2400}{240} = 10.$$

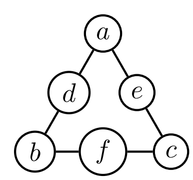
Therefore, the number of multiples of 240 between 1400 and 2400 is $10-5=5$. Thus, the number of multiples of either 15 or 16 (or both) between 1400 and 2400 is:

$$67+63-5 = 125.$$

answer is (c)

Junior Final Part A

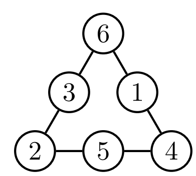
1. Let a, b, c, d, e, and f be the numbers in the circles, as shown. Then by the information given in the problem



$$a+c+d = 11,$$
$$a+b+e = 11,$$
$$b+c+f = 11,$$
$$a+b+c+d+e+f = 21.$$

Adding the first three of these equations gives:

$$2(a+b+c)+d+e+f = 33.$$



Then subtracting the last equation from this give:

$$a+b+c = 33-21 = 12.$$

The only possible placement of the numbers, except for rotations or reflections, is shown.

answer is (b)

2. A number is divisible by 6 if it is even and its digits add up to a multiple of 3. Thus, the first date in each month that is divisible by 6 is: 20030106, 20030202, 20030304, 20030406, 20030502, 20030604, 20030706, 20030802, 20030904, 20031006, 20031102, and 20031204. Since every month, except February, has either 30 or 31 days, every month except February has exactly five days that are multiples of 6. To see this, note that if the first date is the second of the month, then the sequence of dates is 2, 8, 14, 20, and 26, and if the first date is the sixth of the month, then the sequence of dates is 6, 12, 18, 24, and 30. We must check February separately, but in 2003 the first date is the second, looking at the first sequence above shows that there are five dates in February that are divisible by 6. Thus, there is a total of 60 dates in 2003 that are divisible by 6 when written in standard SI form.

Remark: One might also ask whether it is possible to get 6 such dates in a month. To do this, however, requires that we start with the first of the month, and such a date generates an odd number in standard SI form, and we have observed that to be divisible by 6 the number must be even.

answer is (e)

3. Generate the first six terms in the sequence

$$a_1 = 7 \qquad a_2 = \sqrt{|a_1^2 - 16|} = \sqrt{|49-16|} = \sqrt{33}$$
$$a_3 = \sqrt{|a_2^2 - 16|} = \sqrt{|33-16|} = \sqrt{17} \qquad a_4 = \sqrt{|a_3^2 - 16|} = \sqrt{|17-16|} = 1$$
$$a_5 = \sqrt{|a_4^2 - 16|} = \sqrt{|1-16|} = \sqrt{15} \qquad a_6 = \sqrt{|a_5^2 - 16|} = \sqrt{|15-16|} = 1$$

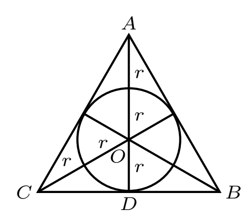
Thus, for $n \geq 7$

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ \sqrt{15} & \text{if } n \text{ is odd} \end{cases} .$$

Therefore, $a_{80} = 1$.

answer is (a)

4. The diagram shows the equilateral triangle ABC with a circle of radius r inscribed with centre at O. The line from vertex A to the midpoint D of the opposite side is a median of the triangle. This median goes through the centre of the circle O, which divides the median AD into segments in the ratio 2 : 1. Thus, the segments AO, BO, and CO all have length $2r$, as shown in the diagram. By Pythagoras' Theorem, the length of segment CD is $\sqrt{3}r$. Therefore, the area of the triangle is



$$A_\triangle = \frac{1}{2} \cdot 3r \cdot 2\sqrt{3}r = 3\sqrt{3}r^2.$$

and, of course, the area of the circle is $A_\bigcirc = \pi r^2$, so the ratio of the area of circle to the area of the triangle is

$$\frac{A_\bigcirc}{A_\triangle} = \frac{\pi}{3\sqrt{3}}.$$

answer is (c)

5. Combining terms gives

$$3^{17} + 3^{17} + 3^{17} + 3^{17} + 3^{17} + 3^{17} + 3^{17} + 3^{17} + 3^{17} = 9 \cdot 3^{17} = 3^2 \cdot 3^{17} = 3^{19}.$$

But, $81^N = \left(3^4\right)^N = 3^{4N}$. Then, $3^{19} = 3^{4N} \implies 19 = 4N \implies N = \dfrac{19}{4}$.

answer is (d)

6. Since the circumference of the small circle is 4π, the central angle in the large circle between successive points of contact of the point A with the large circle is

$$\theta = \frac{s}{r} = \frac{4\pi}{9}.$$

The radius of the circle followed by the centre of the small circle is 11. Thus, the distance travelled by the centre of the small circle before the point A next comes in contact with the large circle is

$$s = r\theta = 11\left(\frac{4\pi}{9}\right) = \frac{44\pi}{9}.$$

answer is (b)

7. We are looking for the least common multiple of pairs of numbers $[a, b]$ (that is, the smallest number divisible by both a and b), where a is in the set $\{8, 9, 15\}$ and b is in the set $\{7, 10, 12\}$. There are nine possible pairs. The smallest common multiple of each pair is:

$$\begin{array}{ccc} [8,7] = 56 & [9,7] = 63 & [15,7] = 105 \\ [8,10] = 40 & [9,10] = 90 & [15,10] = 30 \\ [8,12] = 24 & [9,12] = 36 & [15,12] = 60 \end{array}$$

The smallest value above is 24.

answer is (b)

8. A common multiple of $14a^7b^5c^4 = 2\cdot 7a^7b^5c^4$ and $98a^3b^{15}d^7 = 2\cdot 7^2a^3b^{15}d^7$ is of the form $2^\alpha\cdot 7^\beta\cdot a^\gamma\cdot b^\delta\cdot c^\epsilon\cdot d^\phi$ where $\alpha \geq 1$, $\beta \geq 2$, $\gamma \geq 7$, $\delta \geq 15$, $\epsilon \geq 4$, and $\phi \geq 7$. The product of the two numbers is

$$2^2 \cdot 7^3 \cdot a^{10} \cdot b^{20} \cdot c^4 \cdot d^7.$$

Therefore, if the common multiple divides the product, we must have

$$\alpha \leq 2,\ \beta \leq 3,\ \gamma \leq 10,\ \delta \leq 20,\ \epsilon \leq 4,\ \phi \leq 7$$

Thus, $\alpha = 1$ or 2 (two choices), $\beta = 2$ or 3 (two choices), $\gamma = 7, 8, 9,$ or 10 (4 choices), $\delta = 15, 16, 17, 18, 19,$ or 20 (6 choices), $\epsilon = 4$ (1 choice), and $\phi = 7$ (1 choice). So the total number of possibilities is $2 \cdot 2 \cdot 4 \cdot 6 = 96$.

answer is (b)

9. We must find the three positive integers a, b, and c for which $a + b + c = 14$. Since the triangle is scalene, we may assume that $a < b < c$. (Any permutation of the numbers gives the same triangle.) For a, b, and c to be the sides of a triangle we must have $c < a + b$. The triples of positive integers that add up 14 and satisfy the first inequality are $(1, 2, 11)$, $(1, 3, 10)$, $(1, 4, 9)$, $(1, 5, 8)$, $(1, 6, 7)$, $(2, 3, 9)$, $(2, 4, 8)$, $(2, 5, 7)$, $(3, 4, 7)$, and $(3, 5, 6)$. All of these triples, except the last, do not meet the requirement for a triangle, $c < a + b$. This leaves $(3, 5, 6)$. Thus, the shortest possible side is 3.

answer is (c)

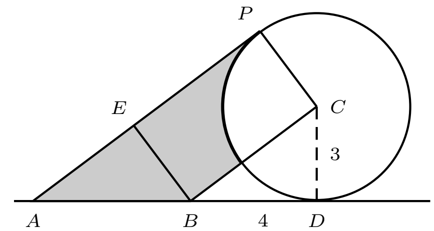
10. The area between the outermost circle (radius is c) and the innermost circle (radius is a) is $\pi c^2 - \pi a^2 = \pi c^2 - 64\pi$. The area between the outermost circle and middle circle (radius is b) is $\pi c^2 - \pi b^2 = \pi c^2 - 81\pi$. Since the middle circle bisects the area between the other two circles, we have

$$\frac{\pi c^2 - 64\pi}{2} = \pi c^2 - 81\pi \quad\Longrightarrow\quad c^2 - 64 = 2c^2 - 162 \quad\Longrightarrow\quad c^2 = 98 \quad\Longrightarrow\quad c = \sqrt{98} = 7\sqrt{2}.$$

answer is (a)

Junior Final Part B

1. On the diagram draw the line segment PC and the line segment from B perpendicular to AP meeting AP at E. Then, $PC = BE = r = 3$. By Pythagoras' Theorem $BC = 5$. Since triangles AEB and BDC are right triangles with $\angle BAE = \angle CBD$ and sides $CD = BE = r = 3$, the two triangles are congruent. Hence, $AB = BC = 5$. So



$$\text{area } \triangle AEB = \tfrac{1}{2} \cdot 4 \cdot 3 = 6,$$
$$\text{area } \square BEPC = 3 \cdot 5 = 15.$$

Finally, the area of the shaded region is the sum of the areas of triangle AEB and rectangle $BEPC$ minus the one quarter of the area of the circle, since angle PCB is a right angle. Thus,

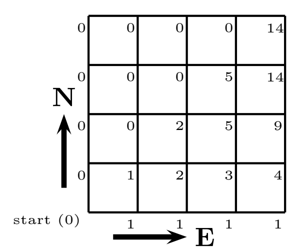
$$\text{shaded area} = 6 + 15 - \frac{1}{4}\pi \cdot 3^2 = 21 - \frac{9\pi}{4}.$$

2. The powers of 2 that are less than the maximum display on the clock, namely 1259, are 0001, 0002, 0004, 0008, 0016, 0032, 0064, 0128, 0256, 0512, and 1024. On a 12-hour clock no 00 hour occurs, so that the only powers of 2 that Marvin sees are 0128, 0256, 0512, and 1024. The order in which Marvin sees these numbers is 1024, 0128, 0256, and 0512. The number of minutes of continuous sleep that Marvin gets are:

$$\begin{array}{rlrcr} 1000 \text{ to } 1024 & \longrightarrow & & & 24\,\text{minutes} \\ 1024 \text{ to } 128 & \longrightarrow & 36 + 2 \cdot 60 + 28 & = & 184\,\text{minutes} \\ 128 \text{ to } 256 & \longrightarrow & 32 + 56 & = & 88\,\text{minutes} \\ 256 \text{ to } 512 & \longrightarrow & 4 + 2 \cdot 60 + 12 & = & 136\,\text{minutes} \\ 512 \text{ to } 700 & \longrightarrow & 48 + 60 & = & 108\,\text{minutes} \end{array}$$

Thus, the maximum number of minutes of continuous sleep that Marvin gets is 184, from 10:24 pm to 1:28 am.

3. The diagram shows the street grid. At each vertex (intersection) we place a number indicating the number of ways to get to the vertex from the lower left corner. All vertices above the diagonal have a zero and all numbers on the bottom edge, except the left-most vertex, have a one. At all other vertices the number is the sum of the numbers immediately west and immediately south of it.



Alternate solution 1: A possible path through the grid can be specified using a string of r's (right, east) and u's (up, north), indicating the direction taken at each of the eight intersections traversed. For the path not to cross the diagonal, reading the string from left to right, there can never be more u's than r's. Enumerate the strings by grouping them according to which intersection on the diagonal is touched first. There are four possibilities: $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. The paths are

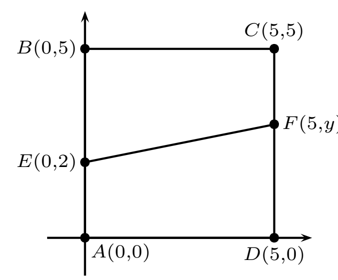
(1,1)	(2,2)	(3,3)	(4,4)
$ru(rururu)$	$rruu(ruru)$	$r(ruru)uru$	$r(rururu)u$
$ru(rurruu)$	$rruu(rruu)$	$r(rruu)uru$	$r(rurruu)u$
$ru(rruuru)$			$r(rruuru)u$
$ru(rruruu)$			$r(rruruu)u$
$ru(rrruuu)$			$r(rrruuu)u$

This gives a total of 14 paths. Note that the parenthesized strings in the $(1, 1)$ and $(4, 4)$ columns represent the five paths in a 3×3 grid, and the parenthesized strings in the $(2, 2)$ and $(3, 3)$ columns represent the two paths in a 2×2 grid.

Alternate solution 2: Using the same notation as in Alternate Solution 1, the total number of paths from home to school without the restriction of not crossing the diagonal is the number of ways putting four r's in the eight possible positions in the string. This can be done in $\binom{8}{4}$ ways. Then we count the number of paths that do cross the diagonal. To do this, consider a path that does cross the diagonal, for example, $rruu\underline{u}rru$. This path crosses the diagonal at $(2, 2)$ and continues upward, as a result of the underlined u. Transform this string by changing every u to r and every r to u after the underlined u. This gives the string $rruuuuur$. This gives a path that ends at the point $(3, 5)$. Any path that goes above the diagonal can be so transformed into a path that ends at $(3, 5)$, since there will be one extra u and one less r. Further, any path that ends at $(3, 5)$, for example $ru\underline{u}ruuur$, can be transformed into a path that ends at $(4, 4)$ and goes above the diagonal, using the same transformation. For the string given, the transformed string is $ruuurrru$, which gives a path that ends at $(4, 4)$ and goes above the diagonal. Thus, the number of paths that go above the diagonal equals the number of paths that end at $(3, 5)$, which is the number of ways of placing three r's in the eight positions in the string. This can be done in $\binom{8}{3}$ ways. Thus, the number of paths is

$$\binom{8}{4} - \binom{8}{3} = 70 - 56 = 14.$$

4. The diagram shows the square with the point E. The line through E that bisects the area of the square intersects the side CD at the point F with coordinates $(5, y)$. Since the line bisects the area of the square, we have:



$$5\left(\frac{2+y}{2}\right) = \frac{25}{2} \implies 2 + y = 5 \implies y = 3.$$

Thus, the line has slope $m = \dfrac{3-2}{5-0} = \dfrac{1}{5}$. So the equation of the line is

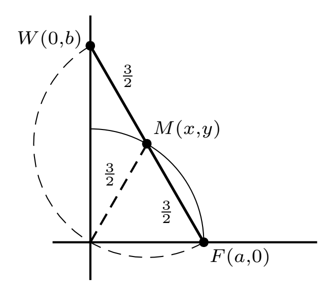
$$y - 2 = \frac{1}{5}(x - 0) \implies y = \frac{1}{5}x + 2.$$

Alternate solution: Any line cutting a square in half must pass through its centre $G\left(\frac{5}{2}, \frac{5}{2}\right)$. Thus, the slope of the line is

$$m = \frac{\frac{5}{2} - 2}{\frac{5}{2} - 0} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{5}{2}\right)} = \frac{1}{5}.$$

Continue as above.

5. The diagram shows the ladder after sliding part way down. Let $F(a, 0)$ be the point where the ladder is in contact with the floor, $W(0, b)$ be the point where ladder is in contact with the wall, and $M(x, y)$ be the midpoint of the ladder. By the midpoint formula



$$x = \frac{a}{2} \quad \text{and} \quad y = \frac{b}{2}.$$

Further, since the ladder is 3 metres long we have

$$a^2 + b^2 = 3 \implies x^2 + y^2 = \frac{a^2 + b^2}{4} = \frac{9}{4},$$

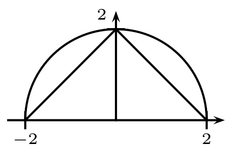
so that the midpoint of the ladder follows a circle with radius $\frac{3}{2}$ centred at the origin. As the ladder slides from the vertical to the horizontal, the midpoint sweeps out a quarter of this circle. Thus, the length of the path followed by the midpoint of the ladder is

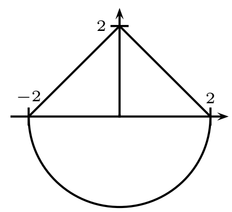
$$\frac{1}{4} \cdot 2\pi r = \frac{1}{4} \cdot 2\pi \cdot \frac{3}{2} = \frac{3\pi}{4} \text{ m}.$$

Alternate solution: Since a right-angled triangle can be inscribed in a semi-circle with the right-angle on the circumference and the hypotenuse along the diameter, we see that the midpoint of the hypotenuse (ladder) is equidistant from all three vertices. Thus, the midpoint of the ladder is $\frac{3}{2}$ metres from the right-angle, where the floor and wall intersect, at all times, that is the midpoint follows a circle of radius $\frac{3}{2}$ centred at the origin. Continue as above.

Senior Final Part A

1. First consider the two possible semi-circles: $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$.





In the figure on the left no area meets the description. On the right, the areas of the triangle above the horizontal axis and semi-circle below are

$$A_\triangle = \frac{1}{2} \cdot 4 \cdot 2 = 4 \quad \text{and} \quad A_\bigcirc = \frac{1}{2}\pi \cdot 2^2 = 2\pi.$$

Thus, the area bounded above by the two lines and below by the semi-circle is

$$A_\triangle + A_\bigcirc = 2\pi + 4.$$

answer is (b)

2. See Problem 8 from the Junior Part A section.

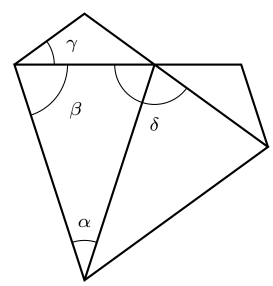
answer is (b)

3. The diagram shows two segments of the regular polygon. The central angle α of the polygon of n-sides is $\frac{360}{n}$, so that the angle β at the base of the interior isosceles triangle is

$$\beta = \frac{1}{2}(180 - \alpha) = 90 - \frac{180}{n}.$$

Thus, the interior angle δ is

$$\delta = 2\beta = 180 - \frac{360}{n}.$$



Then, the base angle of the adjoined isosceles triangle, γ, is

$$\gamma = 180 - \delta = 180 - \left(180 - \frac{360}{n}\right) = \frac{360}{n} = \alpha.$$

Therefore, the vertex angle of the star is

$$180 - 2\gamma = 180 - 2\left(\frac{360}{n}\right) = 180 - \frac{720}{n} = \frac{180n - 720}{n} = \frac{180(n-4)}{n}.$$

answer is (e)

4. Writing the first quadratic equation as $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ shows that

$$mn = \frac{c}{a} \quad \text{and} \quad m + n = -\frac{b}{a}. \tag{$*$}$$

In the same way, the second quadratic equation gives

$$(m+2)(n+2) = \frac{r}{p} = \frac{r}{a} \quad \text{and} \quad (m+2) + (n+2) = -\frac{q}{p} = -\frac{q}{a}. \tag{$**$}$$

Expanding the first equation in (**) and using the equations in (*) gives

$$mn + 2(m+n) + 4 = \frac{r}{a} \quad \Longrightarrow \quad \frac{c}{a} + 2\left(-\frac{b}{a}\right) + 4 = \frac{r}{a} \quad \Longrightarrow \quad c - 2b + 4a = r.$$

Doing the same for the second equality in (**) gives

$$m + n + 4 = -\frac{q}{a} \quad \Longrightarrow \quad -\frac{b}{a} + 4 = -\frac{q}{a} \quad \Longrightarrow \quad -b + 4a = -q \quad \Longrightarrow \quad b - 4a = q.$$

Adding the two equalities above gives

$$q + r = b - 4a + c - 2b + 4a = c - b.$$

answer is (b)

5. Let $\angle NAD = \alpha$ and $\angle MAD = \beta$. By Pythagoras' Theorem, we get

$$\overline{MA} = \overline{NA} = \sqrt{2^2 + 1^2} = \sqrt{5}.$$

Then, $\theta = \beta - \alpha$ so that

$$\begin{aligned}\cos\theta &= \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha \\ &= \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \frac{2}{5} + \frac{2}{5} = \frac{4}{5}.\end{aligned}$$

answer is (d)

6. Any of the functions that are modified by performing an operation on the argument of the function have a different domain than f, and any that are modified by performing an operation on the value of the function have a different range than f. Therefore,

$f(2x)$	has a different domain, but the same range;
$f(x+2)$	has a different domain, but the same range;
$2f(x)$	has the same domain, but different range;
$f\left(\frac{x}{2}\right)$	has a different domain, but the same range;
$\dfrac{f(x)}{2}$	has the same domain, but a different range;
$f(x+2) - 2$	has a different domain, and a different range.

Thus, the number of functions with the same domain is $m = 2$, and the number of functions with the same range is $n = 3$.

answer is (c)

7. The base of the solid is an isosceles right triangle with the two perpendicular sides of length one. Its area is

$$A_b = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}.$$

The top is another isosceles right triangle with the two perpendicular sides of length $\frac{1}{2}$. Its area is

$$A_t = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

Two of the sides are trapezoids cut from the sides that are isosceles triangles. The area of one such side is

$$A_i = \frac{1}{2} \cdot \frac{1}{2} \left(1 + \frac{1}{2}\right) = \frac{3}{8}.$$

The other side is a trapezoid cut from the side that is an equilateral triangle. The longer parallel side of this trapezoid has length $\sqrt{2}$ and the shorter parallel side has length one half of this, namely $\frac{\sqrt{2}}{2}$. The height of this trapezoid is the one half of the distance from the top vertex of the tetrahedron, $(0, 0, 1)$ and the midpoint of the longer side of the base triangle, $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$. So the height of this trapezoid is

$$h = \frac{1}{2}\sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \frac{\sqrt{6}}{4}.$$

and its are is

$$A_e = \frac{1}{2} \cdot \frac{\sqrt{6}}{4} \left(\sqrt{2} + \frac{\sqrt{2}}{2}\right) = \frac{1}{2} \cdot \frac{\sqrt{6}}{4} \cdot \frac{3\sqrt{2}}{2} = \frac{3\sqrt{12}}{16} = \frac{3\sqrt{3}}{8}.$$

Therefore, the total surface area is

$$A = A_b + A_t + 2A_i + A_e = \frac{1}{2} + \frac{1}{8} + 2 \cdot \frac{3}{8} + \frac{3\sqrt{3}}{8} = \frac{4 + 1 + 6 + 3\sqrt{3}}{8} = \frac{11 + 3\sqrt{3}}{8}.$$

answer is (c)

8. The n^{th} odd number is $2n - 1$, and the sum of the first k positive integers is $\dfrac{k(k+1)}{2}$. So the number of entries in the first k rows in the triangular array shown is $\dfrac{k(k+1)}{2}$. This means that the last entry in the k^{th} row is

$$2\left(\frac{k(k+1)}{2}\right) - 1 = k^2 + k - 1,$$

and the first entry in row $k + 1$ is $k^2 + k - 1 + 2 = k^2 + k + 1$. So the first entry in the 43^{rd} row is

$$42^2 + 42 + 1 = 1807.$$

Then, the 29^{th} entry in this row is 28 positions over, which is $1807 + 2 \cdot 28 = 1863$.

answer is (a)

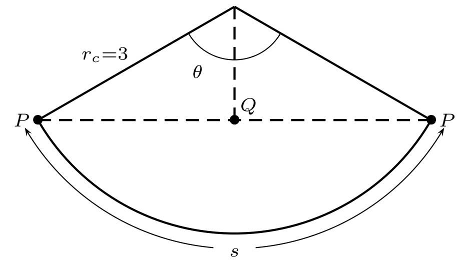
9. Consider all possible ordered sums of the numbers 1, 2, and 3 that give 6:

$$1+2+3 = 1+3+2 = 2+1+3 = 2+2+2 = 2+3+1 = 3+1+2 = 3+2+1 = 6.$$

These seven outcomes are all equally likely and only one has all twos in it. Thus, the probability is $p = \frac{1}{7}$.

answer is (c)

10. Cut the cone along a straight line from its vertex to the point P and flatten it out. The result is a circular segment of radius $r_c = 3$. The shortest distance around the cone is the straight line distance between the two images of P. Let θ be the central angle of the circular sector. The arc-length s of this sector is the circumference of the base of the cone, whose radius is $r_b = 1$. So the arc-length is $s = 2\pi r_b = 2\pi$. Thus,



$$\theta = \frac{s}{r_c} = \frac{2\pi}{3} = 120°.$$

Let Q be the midpoint between the two images of P. Join Q to the vertex of the cone. The two right triangles formed are 30°-60°-90°. Thus

$$\overline{PQ} = \frac{\sqrt{3}}{2} \cdot 3 \quad \Longrightarrow \quad \text{shortest distance} = 2 \cdot \overline{PQ} = 3\sqrt{3}.$$

answer is (d)

Senior Final Part B

1. Starting from the letter T in the centre of the bottom row, label each letter outwardly with a number representing the number of ways we can reach that letter in a spelling of TRIANGLE. For example, each R is labelled with 1, as is each letter in the bottom row. Each other letter has a label which is the sum of the labels of the previous letter of TRIANGLE which are adjacent to it. The labels are shown in the diagram.

1
7 1 7
21 6 1 6 21
35 15 5 1 5 15 35
35 20 20 4 1 4 10 20 35
21 15 10 6 3 1 3 6 10 15 21
7 6 5 4 3 2 1 2 3 4 5 6 7
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

We want the sum of the of numbers associated with the letter E:

$$1 + 7 + 21 + 35 + 35 + 221 + 7 + 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 255.$$

Alternate solution: A possible spelling of TRIANGLE from the right half of the diagram can be specified by giving a string of seven directions r (right) or u (up). There are $2^7 = 128$ such strings. The same is true for the left half of the diagram. Adding these two counts the TRIANGLE up the centre twice. So the number of ways to spell TRIANGLE is $128 + 128 - 1 = 255$.

2. Consider Mary, Pat, and Erin as a single entry in a 10-person race and there are 10 possible positions for this entry in the race. For each of these possibilities there are $3! = 6$ ways to arrange Mary, Pat, and Erin among themselves. This gives a total of $6 \cdot 10 = 60$ possible finishes.

3. See Problem 5 from the Junior Part B section.

4. a.

$$\frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \frac{1}{9\cdot 11} = \frac{7\cdot 3\cdot 11 + 9\cdot 11 + 5\cdot 11 + 5\cdot 7}{5\cdot 7\cdot 9\cdot 11}$$
$$= \frac{231 + 99 + 55 + 35}{5\cdot 7\cdot 9\cdot 11}$$
$$= \frac{4}{33}.$$

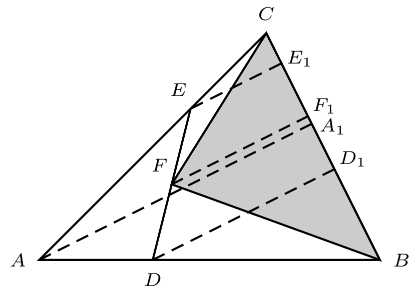
b.

$$\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\frac{1}{9\cdot 11}+\cdots+\frac{1}{(2n+1)(2n+3)}$$
$$=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\frac{1}{2}\left(\frac{1}{7}-\frac{1}{9}\right)+\cdots+\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right),$$

a telescoping sum,

$$=\frac{1}{2}\cdot\frac{1}{3}-\frac{1}{2}\left(\frac{1}{2n+3}\right)=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right)=\frac{1}{2}\left(\frac{2n}{3(2n+3)}\right)=\frac{n}{3(2n+1)}.$$

5. Let $\left[XYZ\right]$ represent the area of triangle XYZ. Draw lines from A, D, F and E to BC, perpendicular to BC and label the points of intersection A_1, D_1, F_1, and E_1. By similar triangles:



$$DD_1=\frac{2}{3}AA_1 \quad\text{and}\quad EE_1=\frac{1}{3}AA_1.$$

Since F is the midpoint of DE, we have

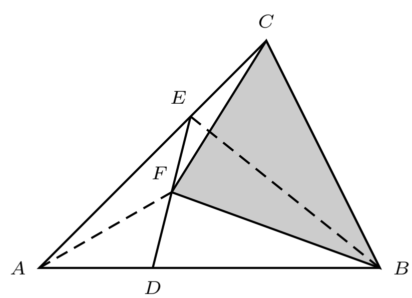
$$FF_1=\frac{1}{2}\left(DD_1+EE_1\right)=\frac{1}{2}\left(\frac{2}{3}AA_1+\frac{1}{3}AA_1\right)=\frac{1}{2}AA_1.$$

Thus, the area of the shaded region, triangle BCF, is

$$\left[BCF\right]=\frac{1}{2}CB\cdot FF_1=\frac{1}{2}\left(\frac{1}{2}CB\cdot AA_1\right)=\frac{1}{2}\left[ABC\right]=\frac{1}{2}.$$

Hence, the area of the shaded region is $\frac{1}{2}$.

Alternate solution: Join A to F and E to B. Now



$$\left[AFE\right]=2\left[EFC\right],$$

since they have the same altitudes ($\perp$ distance between F and AC) and their respective bases, EA and EC, satisfy $EA=2EC$. Using similar arguments we also have:

$$\left[FDB\right]=2\left[AFD\right],\qquad \left[AFD\right]=\left[AFE\right],$$

and

$$\left[AEB\right]=\frac{2}{3}\left[ABC\right],\qquad \left[AEB\right]=2\left[AFB\right].$$

This yields

$$\left[FDB\right]=4\left[EFC\right]\qquad\text{and}\qquad\left[AFB\right]=\frac{1}{3}.$$

But

$$\left[AFB\right]=\left[AFD\right]+\left[DFB\right]=2\left[EFC\right]+4\left[EFC\right]=6\left[EFC\right].$$

Therefore, $\left[EFC\right]=\frac{1}{18}$ and so

$$\text{shaded area }=1-\left[EFC\right]-\left[ADE\right]-\left[DFB\right]=1-\frac{1}{18}-\frac{4}{18}-\frac{4}{18}=\frac{1}{2}.$$