BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2014

Solutions – Draft 1

Junior Preliminary

1. Rearrange the sum as

$$\begin{aligned}(2014+2012+2010+\cdots+2)&-(2013+2011+2009+\cdots+1)\\&=(2014-2013)+(2012-2011)+\cdots+(2-1)\\&=\underbrace{1+1+\cdots+1}_{1007\text{ terms}}=1007\end{aligned}$$

Answer is (D).

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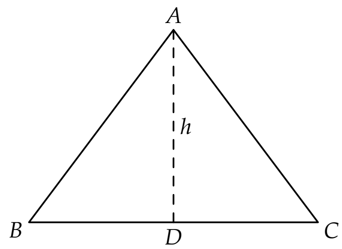
2. Let h be the height of the triangle with BC as base. (See the diagram.) Then, since the area of the triangle is 48, it follows that

$$\tfrac{1}{2}\times h\times BC=\tfrac{1}{2}\times h\times 12=6h=48\Rightarrow h=8$$

Using Pythagoras' theorem in triangle ABD gives

$$AB^2=h^2+BD^2=8^2+6^2=64+36=100\Rightarrow AB=10$$



Answer is (A).

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3. Let n be the middle integer in the set of 25 integers. Then the 25 integers are $n-12$, $n-11$, $\ldots$, n, $\ldots$, $n+11$, $n+12$, and their sum is

$$(n-12)+(n-11)+\cdots+n+\cdots+(n+11)+(n+12)=25n$$

Since this sum is equal to 50, then

$$25n=50\Rightarrow n=2$$

Therefore, the largest integer is $n+12=14$.

Answer is (C).

4. Counting from the top of the pyramid the number of blocks on level n is

$$L_n = 1 + 2 + 3 + \cdots + n$$

So the number of blocks on the ten levels are:

$$\begin{aligned} L_1 &= 1 \\ L_2 &= 1 + 2 = 3 \\ L_3 &= 1 + 2 + 3 = 6 \\ &\vdots \\ L_9 &= 1 + 2 + 3 + \cdots + 9 = 45 \\ L_{10} &= 1 + 2 + 3 + \cdots + 9 + 10 = 55 \end{aligned}$$

So the total number of blocks is

$$C = 1 + 3 + 6 + \cdots + 45 + 55 = 220$$

Note that another approach is to add the sums above in columns to give

$$C = 10\,(1) + 9\,(2) + 8\,(3) + \cdots + 3\,(8) + 2\,(9) + 1\,(10) = 20 + 36 + 48 + 56 + 60 = 220$$

Answer is (A).

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5. The intercepts occur when $x + 1 = 0 \Rightarrow x = -1$ and $x - 3 = 0 \Rightarrow x = 3$. The distance between them is $3 - (-1) = 4$.

Answer is (E).

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6. The average age of the combined group is:

$$\text{combined average} = \frac{40 \times 12 + 60 \times 40}{40 + 60} = \frac{40\,(12 + 60)}{100} = \frac{4 \times 72}{10} = \frac{288}{10} = 28.8$$

Answer is (B).

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7. Putting two items in each box uses 16 objects. Adding any further objects requires that at least one box has at least three objects. So the smallest number of objects that ensures at least one of the 8 boxes has at least three objects is 17.

Answer is (D).

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8. Evidently, one man can dig one hole in five hours. Since $200 = 4 \times 50$, it will take 50 men $4 \times 5 = 20$ hours to dig 200 holes.

Alternative solution:

From the given information it takes 25 worker hours to dig 5 holes. So that 5 worker hours are required to dig 1 hole. Hence, for 200 holes a total of 1000 worker hours is required. Dividing by 50 workers shows that each worker has to work 20 hours.

Answer is (C).

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9. Since three cells can be found such that any two of them have a side in common, it is clear that two colours will not be sufficient. The diagram shows a colouring using 3 colours that meets the stated requirements. Hence, the minimum number of colours required is 3.

Answer is (B).

10. Consider the cases

- for $0 \leq n \leq 99$, each group of ten has a 4 in the unit's place, and for $40 \leq n \leq 49$ there are ten 4's in the ten's place. Giving a total of $10 + 10 = 20$.
- for $0 \leq n \leq 999$, each group of one hundred has 20 between the unit and tens places, by part (a); in addition, for $400 \leq n \leq 490$ there are one hundred 4's in the hundreds place. Giving a total of $10 \times 20 + 100 = 300$.
- then for $0 \leq n \leq 999$ there are 300; for $1000 \leq n \leq 1999$ there are 300; and for $2000 \leq n \leq 2014$ there are 2.

This gives a total of $300 + 300 + 2 = 602$.

Alternative solution:

First, consider the numbers 0000, 0001, 0002, ..., 1998, 1999 and note that the digit '4' never appears in the first position. So there are $3 \times 2000 = 6000$ other digits to consider. Exactly 1/10 of these will be 4's, giving 600 appearances of the digit 4 in these numbers. Further, there is one 4 in each of 2004 and 2014, giving a total of 602 times that the digit 4 appears.

Answer is (D).

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11. The lines drawn on the diagram partition the rectangle into 8 congruent right triangles. Let θ the smaller of the non-right angles in the triangles. Then, by symmetry the side opposite the angle θ has length 4. By Pythagoras' theorem the length of the other is

$$\ell = \sqrt{8^2 - 4^2} = 4\sqrt{3}$$

Hence,

$$\text{area of one triangle} = \tfrac{1}{2} \times 4 \times 4\sqrt{3} = 8\sqrt{3}$$

so that

$$\text{area of rectangle} = 8\left(8\sqrt{3}\right) = 64\sqrt{3}$$

Answer is (B).

12. The number of ways to choose the four numbers from the given set of ten distinct numbers, without replacement, is

$$ {}_{10}C_4 = \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210 $$

The only way that the product can be odd is if each of the numbers is odd. There are 5 odd numbered balls. The number of ways to choose four of these five odd numbers is

$$ {}_{5}C_4 = \frac{5!}{1!4!} = 5 $$

Hence the probability is

$$ \frac{5}{210} = \frac{1}{42} $$

Alternative solution:

The probability of getting an odd numbered ball on all four draws is the *product* of the probabilities of getting an odd numbered ball on each draw, keeping in mind that one odd numbered ball has been removed on the previous draw. Hence, the probability is

$$ \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{1}{42} $$

Answer is (E).

Senior Preliminary

1. For $n = 5$ the value of the product is

$$ \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} $$

Cancelling common factors gives

$$ \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right) = \frac{7 \cdot 9}{2^8} = \frac{63}{256} $$

Answer is (E).

2. Since three cells can be found such that any two of them have a side in common, it is clear that two colours will not be sufficient. The diagram shows a colouring using 3 colours that meets the stated requirements. Hence, the minimum number of colours required is 3.

Answer is (B).

3. Let r be the radius of the pizza. Let R be the radius of the pan and D be the centre of the pan. By symmetry the radius AD of the pan makes a 45° angle with the radius AB of the pizza. Since both A and B are points on the circumference of the pan, then $AD = BD = R$. Thus, triangle ADB is an isosceles right triangle with right angle at D. Hence, $AB = r = \sqrt{2}R$ and

$$ \text{area of pizza slice} = \tfrac{1}{4}\pi r^2 = \tfrac{1}{4}\pi\left(\sqrt{2}R\right)^2 = \tfrac{1}{2}\pi R^2 $$

This is one-half of the area of the pan.

Answer is (B).

4. These parabolas have the same axis of symmetry (the y-axis in this case), and one opens upward with vertex at $(0, 2)$ and the other opens downward with vertex at $(0, -4)$. Thus, the shortest distance between them is the distance between the vertices, which is 6.

Answer is (C).

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5. In each of the ranges 1 to 9, 10 to 19, 20 to 29, 50 to 59, 60 to 69, 70 to 79, 80 to 89, and 90 to 100 there are two integers containing a 3 or a 4. In each of the ranges 30 to 39, inclusive, and 40 to 49, inclusive, there are ten integers containing a 3 or a 4. Hence, the total number of such integers is $(8 \times 2) + (2 \times 10) = 36$.

Answer is (A).

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6. Write the second equation as $Ax^2 + Bx - 1 = 0$. Then multiply the first equation by $-\frac{1}{5}$ to give

$$-\tfrac{1}{5}x^2 + \tfrac{6}{5}x - 1 = 0$$

This matches the second equation if $A = -\frac{1}{5}$ and $B = \frac{6}{5}$. Then

$$A + B = -\tfrac{1}{5} + \tfrac{6}{5} = 1$$

Answer is (D).

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7. It is required to count how many times 2×5 appears in 2014!. First note that the factor 2 appears far more frequently than that of 5 so it is only necessary to count the number of times that 5 appears. Now dividing 2014 by successive powers of 5 gives

$$\begin{aligned} 2014 &= 5 \times 402 + 4, \\ &= 25 \times 80 + 14, \\ &= 125 \times 16 + 14, \\ &= 625 \times 3 + 139, \end{aligned}$$

Thus, there are $402 + 80 + 16 + 3 = 501$ factors of 5's in 2014!. (Note that it is not correct to conclude that the number of times that 5 appears is given by $402 + 2 \times 80 + 3 \times 16 + 4 \times 3$, since this results in multiple counts of lower powers of 5.)

Answer is (B).

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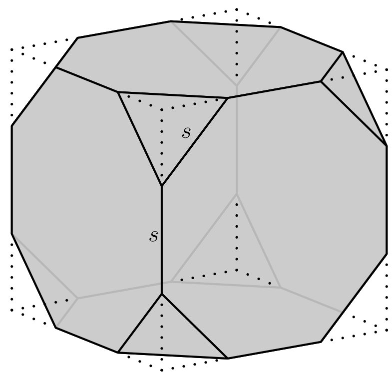
8. Since ABC is an isosceles triangle with $AB = AC$ and $\angle BAC = 40^\circ$, then $\angle ABC = \angle ACB = 70^\circ$. Since $BC = CE$, then $\angle BEC = \angle ABC = 70^\circ$, so that $\angle BCE = 40^\circ$. Thus, $\angle DCE = 30^\circ$, and, since $CD = DE$, CDE is an isosceles triangle with $\angle CED = \angle DCE = 30^\circ$. Therefore, $\angle CDE = 120^\circ$ and so $\angle ADE = 60^\circ$. (Or note that $\angle AED = 80^\circ \Rightarrow \angle ADE = 60^\circ$.) See the diagram.

Answer is (D).

9. A cube has 12 edges. To obtain the maximum number of edges, a triangle is formed at each of the 8 vertices of the cube, while still retaining a part of each edge of the original cube. So the number of edges of the resulting solid is $12 + (8 \times 3) = 36$. The resulting solid is shown.



Answer is (C).

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10. Every two digit number is of the form $10a + b$. There are $5^2 = 25$ such numbers using only odd digits, since there are five non-zero choices (1, 3, 5, 7, 9) for each of the ones and tens columns. Hence, each of the digits 1, 3, 5, 7, 9 appears 5 times, and the sum of all the numbers is

$$10 \times 5\,(1+3+5+7+9) + 5 \times (1+3+5+7+9) = 5 \times 25 \times 11 = 1375$$

Alternative solution:

The required sum is

$$\underbrace{11 + 13 + \cdots + 19}_{\text{i}} + \underbrace{31 + 33 + \cdots + 39}_{\text{ii}} + \cdots + \underbrace{91 + 93 + \cdots + 99}_{\text{v}}$$

$$= \underbrace{50 + (1+3+\cdots+9)}_{\text{i}} + \underbrace{3 \times 50 + (1+3+\cdots+9)}_{\text{ii}} + \cdots + \underbrace{9 \times 50 + (1+3+\cdots+9)}_{\text{v}}$$

$$= 50\,(1+3+\cdots+9) + 5\,(1+3+\cdots+9)$$

$$= 55\,(1+3+\cdots+9) = 55 \times 25 = 1375$$

Note that the average of the groups of five numbers are: 15, 35, 55, 75, and 95. The average of these five numbers is 55. This is the average of all of the numbers. Therefore, the value of the sum of the 25 two digit numbers made from only odd digits is $25 \times 55 = 1375$.

Answer is (A).

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11. Rewriting the second equation gives

$$\frac{b^2 + a^2}{a^2 b^2} = m \Rightarrow a^2 + b^2 = a^2 b^2 m = (ab)^2\, m$$

Since $ab = k$ this can be written as $a^2 + b^2 = k^2 m$. Now

$$(a-b)^2 = a^2 - 2ab + b^2 = \left(a^2 + b^2\right) - 2\,(ab) = k^2 m - 2k = k\,(km - 2)$$

Answer is (E).

12. Let

$$S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} + \cdots$$

Then, writing the sum as

$$\begin{aligned} S \;=\; & \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^n} + \cdots \\ & \qquad + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^n} + \cdots \\ & \qquad\qquad + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^n} + \cdots \\ & \qquad\qquad\qquad \vdots \end{aligned}$$

it is clear that

$$\begin{aligned} S &= 1 + \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^n} + \cdots\right) + \tfrac{1}{2^2}\left(\tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^n} + \cdots\right) + \cdots \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^n} + \cdots = 2 \end{aligned}$$

Alternative solution:
Write the sum as

$$\begin{aligned} S &= \tfrac{1}{2} + \tfrac{2}{2^2} + \tfrac{3}{2^3} + \cdots + \tfrac{n}{2^n} + \cdots \\ &= \tfrac{1}{2} + \tfrac{1+1}{2^2} + \tfrac{1+2}{2^3} + \cdots + \tfrac{1+(n-1)}{2^n} + \cdots \\ &= \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots + \tfrac{1}{2^n} + \cdots + \tfrac{1}{2^2} + \tfrac{2}{2^3} + \cdots + \tfrac{n-1}{2^n} + \tfrac{n}{2^{n+1}} + \cdots \\ &= 1 + \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{2}{2^2} + \tfrac{3}{2^3} + \cdots + \tfrac{n-1}{2^{n-1}} + \tfrac{n}{2^n} + \cdots\right) = 1 + \tfrac{1}{2}S \end{aligned}$$

Solving for S gives $\frac{1}{2}S = 1 \Rightarrow S = 2$. As above.

Answer is (C).

Junior Final, Part A

1. Using Pythagoras' theorem, the length of the other side of the triangle is $\sqrt{15^2 - 9^2} = \sqrt{144} = 12$. So the area of the triangle is $\frac{1}{2}(9 \times 12) = 54$. The area of the rectangle is $9 \times 15 = 135$. Hence, the area of the shaded region is $135 - 54 = 81$.

Answer is (B).

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2. To start the subtraction, borrow 1 from the 10's place of the 2014. This leaves a 0 in the 10's place. Then borrow a 1 from the 100's place. At this point observe that $b = 10 - a \Rightarrow a + b = 10$. The possibilities for a and b are: $4 + 6 = 10$, $3 + 7 = 10$, $2 + 8 = 10$, or $1 + 9 = 10$. Since $a < b$, the only possibility for which neither a nor b equals another digit is $a = 3$ and $b = 7$.

Answer is (C).

3. Let R, r, and j be the percentages of rock, rap, and jazz, after the jazz is added. Then

$$\frac{R}{r} = \frac{68}{32} = \frac{17}{8} \Rightarrow R = \left(\frac{17}{8}\right) r \text{ and } j = 25$$

and

$$R + r + j = 100 \Rightarrow \left(\frac{17}{8}\right) r + r = \left(\frac{25}{8}\right) r = 75 \Rightarrow r = 24$$

Thus, after the jazz is added 24% of the playlist is rap.

Alternative solution:

After the jazz is added the rap and rock together make up 75% of the playlist. Therefore, the percentage of rap in the playlist after the jazz is added will be $0.75 \times 32 = 24$.

Answer is (B).

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4. The total number of games played in the tournament is

$$\text{number of games} = \frac{4 \times 480}{20} = 96$$

So the number of games supervised by each of the 16 coaches is $\frac{96}{16} = 6$.

Answer is (E).

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5. Perform the multiplication as

$$\begin{aligned} 123\,456\,789 \times 999\,999\,999 &= 123\,456\,789\,(1\,000\,000\,000 - 1) = 123\,456\,789\,000\,000\,000 - 123\,456\,789 \\ &= 123\,456\,788\,999\,999\,999 - 123\,456\,789 + 1 = 123\,456\,788\,876\,543\,211 \end{aligned}$$

So the number of 8's appearing in the answer is 3.

Answer is (D).

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6. Consider the following even numbers:

- 2 is obviously practical.
- 4 is done as an example.
- the divisors of 6 are 1, 2, 3, 6, and $4 = 3 + 1$, $5 = 2 + 3$.
- the divisors of 8 are 1, 2, 4, 8, and $5 = 4 + 1$, $6 = 4 + 2$, $7 = 4 + 3 + 1$.
- the divisors of 10 are 1, 2, 5, 10, and 4 is obviously not the sum of distinct divisors of 10.

The smallest even impractical number is 10.

Answer is (C).

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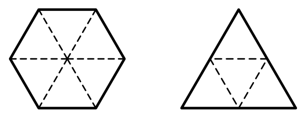
7. Only the three unseen arrows can contain a tip-to-tip meeting. This can happen in three ways, which are mutually exclusive. Each such way of having a tip-to-tip meeting occurs with probability

$$(1/4) \times (1/4) = 1/16$$

so the probability of no tip-to-tip meetings (other than the one shown) is $1 - 3/16 = 13/16$.

Answer is (A).

8. Since the two pieces of wire are of equal length the perimeters of the hexagon and triangle are equal. It is apparent that the side length of the triangle is twice the side length of the hexagon. Hence, the triangle and hexagon can be divided into congruent equilateral triangles, as shown in the diagram. There are 6 triangles in the hexagon and 4 in the triangle. Therefore, if the area of the triangle is 4, then the area of the hexagon is 6. So, if the area of the triangle is 2, then the area of the hexagon is 3.



Answer is (C).

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9. The total distance that Ronnie travels is $5 \times 100 = 500$ kilometres. The total time taken to travel through the towns is $5 \times 5 = 25$ minutes. So the total distance traveled through the towns is

$$\frac{25}{60} \times 60 = 25\,\text{km}$$

Hence, the total distance Ronnie travels on the highway, when not driving through a town, is $500 - 25 = 475\,$km. The time, in hours, that Ronnie travels on the highway when not driving through a town is

$$5 - \frac{25}{60} = \frac{55}{12}$$

Therefore, Ronnie's average highway speed when not driving through a town is

$$\frac{475}{(55/12)} = \frac{19 \times 60}{11} = \frac{1140}{11} = 103\tfrac{7}{11}$$

Answer is (A).

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10. Let $2a$ be the side length of each square and $\theta = \angle BOE$ (see the diagram). Then

$$\theta + \angle AOB + \angle AOF = 180^\circ$$

But $\angle AOB = 90^\circ$, so that $\angle AOF = 90 - \theta$. Therefore, $\angle BEO = \angle AOF$. Further, $EO = AF$, so that triangles BEO and OFA are congruent. Hence, $BE = FO$ which means that $BC = AD$. Now, line segment AB divides the shaded region into two right triangles, ABC and AOB. Let $x = BC = AD$. Then, $AC = 2a - x$ and the area of triangle ABC is given by

$$\text{area of } \Delta\, ABC = \tfrac{1}{2}x\,(2a - x) = xa - \tfrac{1}{2}x^2$$

Triangles BOE and OAF are congruent, so that $BO = AO$. Further, $BE = a - x$ so that by Pythagoras' theorem

$$BO^2 = (a - x)^2 + a^2 = 2a - 2xa + x^2 \Rightarrow BO = AO = \sqrt{2a^2 - 2xa + x^2}$$

Thus, the area of triangle AOB is given by

$$\text{area of } \Delta\, AOB = \tfrac{1}{2}\left(2a^2 - 2xa + x^2\right) = a^2 - xa + \tfrac{1}{2}x^2$$

Therefore, the area of the shaded region is given by

$$\text{area of } \Delta\, ABC + \text{area of } \Delta\, AOB = xa - \tfrac{1}{2}x^2 + a^2 - xa + \tfrac{1}{2}x^2 = a^2$$

The shows that the area does not depend on the orientation of the second square. The total area of each square is $(2a)^2 = 4a^2$. So the area of the shaded region is 1/4 of the area of the leftmost square.

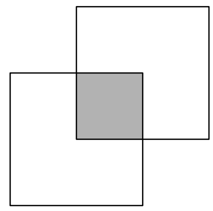
Note that the same solution can be obtained using coordinate geometry.

Problem 10 continued ...

...Problem 10 continued

Alternative solution:

If the problem is well posed, it must be assumed that the area of the shaded region does not depend on the orientation of the rightmost square. If this square is not rotated, the overlap region is square of side a. In this case, the area is a^2 (see the diagram). This is 1/4 of the area of the leftmost square.



Answer is (D).

Junior Final, Part B

1. Possible solutions are

$$2 = 2\times\left(0+1^4\right) = -2+0\times1+4 = 2+0\times(1+4)$$
$$3 = 2+0+1^4 = -2+0+1+4$$
$$4 = 2\times0\times1+4 = 2^{0\times1}\times4$$
$$5 = 2+0-1+4 = 2^{0\times1}+4$$
$$6 = 2+0\times1+4 = 2^0+1+4 = 2^{0+1}+4$$
$$7 = 2+0+1+4$$
$$8 = (2+0\times1)\times4 = 2^{(0-1+4)} = 2^{0+1}\times4$$
$$9 = (2+0+1)^{\sqrt{4}}$$
$$10 = 2\times(0+1+4) = (2+0)\times(1+4)$$

Answer: See the list above.

2. Since the numbers in the diagonal containing the 1 must add to 18, the numbers a and b can only be 8 or 9. Further, for the diagonal containing the 6 and x to add to 18 it must be true that $x+c=12$. The only pair of distinct positive integers less than 8 that add to 12 is 5 and 7. Hence, x and c can only be 5 or 7. Therefore, there are only four cases to consider. The table below summarizes the results:

a	b	c	x	d	e	f	possible
8	9	7	5	5			no
8	9	5	7	3	2	4	yes
9	8	7	5		5	3	no
9	8	5	7		3	5	no

The only case that gives distinct values for the numbers in the circles is the second, in which $x=7$.

Answer: $x=7$.

3. (a) Factoring into prime factors gives

$$49000 = 2^3 \times 5^3 \times 7^2 = 8 \times 125 \times 49$$

Redistributing the prime factors will give at least one pair of factors that have a common factor. Hence, the integers are 8, 125 and 49. Their sum is $8 + 125 + 49 = 182$.

Answer: The sum of the integers is 182.

(b) Any odd integer k is of the form $2n - 1$ where n is an integer. Then

$$k^2 = (2n-1)^2 = 4n^2 - 4n + 1 = 4n(n-1) + 1.$$

Notice that either n or $n - 1$ is even, and therefore $4n(n-1) = 8q$ for some integer q. Thus, the claim holds.

Answer: See the proof above.

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4. (a) Taking the numbers in S in increasing order, the sumset is

$$\begin{aligned} S + S &= \{3+3, 3+5 = 5+3, 3+6 = 6+3, 5+5, 5+6 = 6+5, 6+6\} \\ &= \{6, 8, 9, 10, 11, 12\} \end{aligned}$$

This set contains six distinct numbers.

Answer: See proof above.

(b) Note that there is a total of six sums, namely, $a + a = 2a$, $a + b = b + a$, $a + c = c + a$, $b + b = 2b$, $b + c = c + b$, and $c + c = 2c$. The following inequalities apply:

$$\begin{aligned} a < b &\Rightarrow 2a < a + b < 2b \\ b < c &\Rightarrow 2b < b + c < 2c \end{aligned}$$

Hence, $2a < a + b < 2b < b + c < 2c$. Therefore, there are at least five distinct sums $a + a$, $a + b$, $b + b$, $b + c$, and $c + c$.

Answer: See proof above.

(c) From the arguments above, if the sixth sum $a + c$ is not distinct from the others, it must equal $b + b = 2b$, that is, $a + c = 2b$. There are three possibilities:

$$\begin{aligned} a = 15 \;\&\; b = 21 &\Rightarrow c = 42 - 15 = 27 \Rightarrow S = \{15, 21, 27\} \\ a = 15 \;\&\; c = 21 &\Rightarrow 2b = 21 + 15 = 36 \Rightarrow b = 18 \Rightarrow S = \{15, 18, 21\} \\ b = 15 \;\&\; c = 21 &\Rightarrow a = 30 - 21 = 9 \Rightarrow S = \{9, 15, 21\} \end{aligned}$$

Thus, the possible values of the third number are 9, 18, or 27.

Answer: The possible values of the third number are 9, 18, or 27.

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5. Consider the triangle AOB in the diagram. Since the minimum vertical distance between the two small coins is 2 and the minimum horizontal distance between the large coins is 6, we have $OB = r + 1$ and $OA = R + 3$. Then applying Pythagoras' theorem to triangle AOB gives

$$(r+1)^2 + (R+3)^2 = (R+r)^2$$

Expanding gives

$$r^2 + 2r + 1 + R^2 + 6R + 9 = R^2 + 2rR + r^2 \Rightarrow r + 3R + 5 = rR \Rightarrow r = \frac{3R+5}{R-1}$$

For example, if the right triangle is 5-12-13, the radii are 4 and 9.

Answer: The expression is $r = (3R+5)/(R-1)$.

Senior Final, Part A

1. See Junior Final Part A Problem 4.

Answer is (E).

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2. See Junior Final Part A Problem 6.

Answer is (C).

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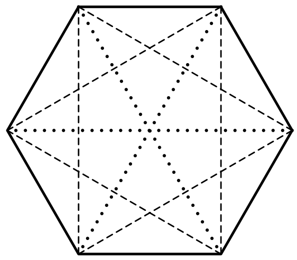
3. There are six sides (solid), three long diagonals (dotted), and six short diagonals (dashed). (See the diagram.) The probability that both the first and the second segments selected are sides is



$$\frac{6}{15} \times \frac{5}{14} = \frac{1}{7}$$

The probability that both the first and the second segments are long diagonals is

$$\frac{3}{15} \times \frac{2}{14} = \frac{1}{35}$$

The probability that both the first and the second segments are short diagonals is

$$\frac{6}{15} \times \frac{5}{14} = \frac{1}{7}$$

Hence, the probability that the two selected segments have the same length is

$$\frac{1}{7} + \frac{1}{35} + \frac{1}{7} = \frac{11}{35}$$

Alternative solution:

There are ${}_6C_2 = 15$ line segments and there are ${}_{15}C_2 = 105$ ways to select two segments without replacement. There are ${}_6C_2 = 15$ ways to select two sides, ${}_3C_2 = 3$ ways to select two long diagonals, and ${}_6C_2 = 15$ ways to select two short diagonals. Hence, the number of ways to select two edges with the same length is $15 + 15 + 3 = 33$. Therefore, the probability that two selected segments have the same length is

$$\frac{33}{105} = \frac{11}{35}$$

Answer is (E).

4. Every three digit number is of the form $100a + 10b + c$. There are $5^3 = 125$ such numbers using only odd digits, since there are five non-zero choices (1, 3, 5, 7, 9) for each of the ones, tens, and hundreds columns. Hence, each of the digits 1, 3, 5, 7, 9 appears 5 times, and the sum of all the numbers is

$$100 \times 25\,(1+3+5+7+9) + 10 \times 25\,(1+3+5+7+9) + \\ 25 \times (1+3+5+7+9) = 25 \times 25 \times 111 = 69375$$

Note that this is equivalent to 125×555.

Alternative solution:

Using the result from Senior Preliminary Problem 10 the required sum is

$$\underbrace{111 + 113 + \cdots + 199}_{\text{i}} + \cdots + \underbrace{911 + 913 + \cdots + 999}_{\text{v}}$$
$$= \underbrace{25 \times 100 + (11 + 13 + \cdots + 99)}_{\text{i}} + \cdots + \underbrace{25 \times 900 + (11 + 13 + \cdots + 99)}_{\text{v}}$$
$$= 2500\,(1 + 3 + \cdots + 9) + 5\,(11 + 13 + \cdots + 99)$$
$$= 2500 \times 25 + 5\,(1375) = 69375$$

Answer is (D).

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5. Triangle BCD is an isosceles right triangle with legs $BC = CD = 4$. So the hypotenuse $BD = 4\sqrt{2}$. Then triangle ABD is a right triangle with legs $BD = 4\sqrt{2}$ and $AD = 2$. So the hypotenuse is

$$AB = \sqrt{\left(4\sqrt{2}\right)^2 + 2^2} = \sqrt{32 + 4} = 6$$

Answer is (B).

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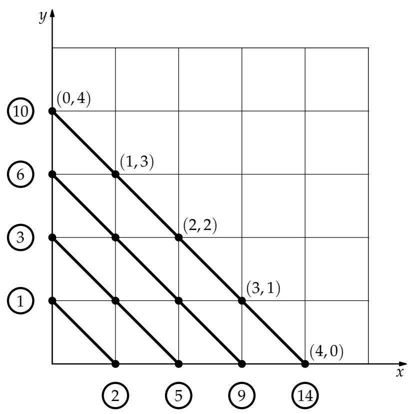
6. The numbering of the points along the horizontal and vertical axes is shown. The number at the n^{th} point on the vertical axis, whose y coordinate is n, is the sum of all of the integers from 1 to n. This is

$$1 + 2 + 3 + \cdots + n = \frac{n\,(n+1)}{2}$$

So to find the coordinates of the 81^{st} point, first find the largest value of n for which

$$\frac{n\,(n+1)}{2} \leq 81$$

Testing values of n shows that for $n = 12$ the value is 78. Hence, the point at $(0, 12)$ is numbered 78. Counting down the line gives the coordinates of the 81^{st} point as $(3, 9)$.



Answer is (A).

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7. A day-knight can only make the statement "it is day on the surface", since he must tell the truth during the day and lie at night. Consequently, the speaker must be a night-knight, and the statement is false. If it were night then he would have had to tell the truth and he did not. The speaker must be a night-knight and it is day on the surface.

Answer is (C).

8. See Junior Final Part A Problem 7 solution, but now there is the potential (with probability 1/4) that the right arrow that we see meets the one on the bottom tip-to-tip. Then the probabilities of at least one new tip-to-tip meeting, based on the direction of the bottom arrow are:

- Points right: probability 1 of new tip-to-tip
- Points forward: probability 1/16 of new tip-to-tip (other two unseen arrows)
- Points left: probability 1/4 of new tip-to-tip (involving it) + 1/16 (other two)
- Points back: probability 1/4 of new tip-to-tip (involving it) + 1/16 (other two)

Overall probability of another tip-to-tip is then

$$\frac{1}{4} + \frac{1}{64} + \frac{1}{16} + \frac{1}{64} + \frac{1}{16} + \frac{1}{64} = \frac{30}{64} = \frac{27}{64}$$

so the desired probability (of the visible one being the only one) is

$$1 - \frac{27}{64} = \frac{37}{64}$$

Answer is (A).

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9. In the diagram the radius of the second circle is $r = AP$. Since $OA = OB = AB = 2$, triangle ABO is an equilateral triangle, so that $\angle AOC = 30^\circ$. Hence, AOC is a 30°-60°-90° triangle with hypotenuse $AO = 2$. The side opposite the 60° angle is $CO = \sqrt{3}$, and the side opposite the 30° angle is $AC = 1$. Thus, $CP = 2 - \sqrt{3}$ and, by Pythagoras' theorem,

$$AP = \sqrt{AC^2 + CP^2} = \sqrt{1 + \left(2 - \sqrt{3}\right)^2} = \sqrt{1 + \left(4 - 4\sqrt{3} + 3\right)} = \sqrt{8 - 4\sqrt{3}} = 2\sqrt{2 - \sqrt{3}}$$

Answer is (B).

---

10. With 4 colours the centre cell, cell a, can be coloured in 4 ways. No other cell can have this colour. Then, one of the other cells, say cell b, can be coloured in 3 ways. After this there are three cases to consider for the remaining cells:

- No other cell is the same colour as cell b. There are 2 ways to colour cell c, and once this cell is coloured the colours alternate around the honeycomb. This gives 2 ways to colour the remaining cells.
- Only one other cell is the same colour as cell b. There are 3 ways to select the cell that is the same colour as cell b, these are cells d, e, and f. Whichever cell is chosen divides the remaining cells into two disjoint sets. For each of these disjoint sets choosing one of the two remaining colours for one cell in the set determines the colours in all of the cells in the set. This gives a total of $3 \times 2 \times 2 = 12$ ways to colour the remaining cells.
- Two other cells are the same colour as cell b. These are cells d and f. This leaves three disjoint cells to colours, each of which can be coloured in 2 ways. This gives $2 \times 2 \times 2 = 8$ ways to colour the remaining cells.
- It is not possible to have more than two other cells the same colour as cell b without having two adjacent cells with the same colour.

Therefore, the number of ways to colour the honeycomb with four colours is:

$$4 \times 3 \times (2 + 12 + 8) = 12 \times 22 = 264$$

Answer is (D).

Senior Final, Part B

1. See Junior Final Part B Problem 2.

Answer: $x = 7$

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2. Let p and q be twin primes with $p = q + 2 > 5$. The integer between q and p is $q + 1$, and it is required to show that $q + 1$ is divisible by 6; that is, $q + 1$ is divisible by both 2 and 3.
Since $p > 5$ and $q = p - 2 > 3$, both $p = q + 2$ and q are odd numbers, and therefore, $q + 1$ is even. Thus, $q + 1$ is divisible by 2.
Since $q > 3$ is prime, the remainder must be either 1 or 2, when q is divided by 3; that is, $q = 3n + 1$ or $q = 3n + 2$ for some integer n. If $q = 3n + 1$, then

$$p = q + 2 = (3n + 1) + 2 = 3n + 3 = 3 \times (n + 1),$$

which contradicts the fact that p is prime. Thus, $q \neq 3n + 1$ and $q = 3n + 2$. Thus

$$q + 1 = (3n + 2) + 1 = 3 \times (n + 1),$$

and therefore $q + 1$ is divisible by 3. It follows that the integer between any twin primes is divisible by 6, provided the larger prime is greater than 5 (or the smaller prime greater than 3).

Answer: See the proof above.

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3. Since ABC is an isosceles triangle with $AB = AC$ and $\angle BAC = x$, we have

$$\angle ABC = \angle ACB = \tfrac{1}{2}(180 - x) = 90 - \tfrac{1}{2}x$$

Since $BC = CE$, we have

$$\angle BEC = \angle ABC = 90 - \tfrac{1}{2}x \Rightarrow \angle BCE = 180 - 2\left(90 - \tfrac{1}{2}x\right) = x$$

Thus,

$$\angle DCE = \angle ACB - \angle BCE = 90 - \tfrac{1}{2}x - x = 90 - \tfrac{3}{2}x$$

Now $CD = DE$. Then CDE is an isosceles triangle with

$$\angle CED = \angle DCE = 90 - \tfrac{3}{2}x$$

Therefore,

$$\angle CDE = 180 - 2\left(90 - \tfrac{3}{2}x\right) = 3x \Rightarrow \angle ADE = y = 180 - 3x$$

Answer: The angle is $y = 180 - 3x$.

4. Each coin can land with a number showing or not, so there are $2^{10} = 1024$ possibilities. Note that the sum of all the integers from 1 to 10 is $1+2+3+4+5+6+7+8+9+10 = 55 \geq 45$. This gives one possibility for a sum that is at least 45. Now consider the numbers that will not be visible. For the sum of the visible numbers to be at least 45, the sum of these numbers can be at most 10.

   - Any one of the numbers can be can be the lone number that is not visible, giving 10 possibilities.
   - If a pair of numbers is not visible, the possibilities are:
     - The lesser of the pair is 1, then the other can be $2, 3, 4, 5, 6, 7, 8, 9$, giving 8 possibilities.
     - The lesser of the pair is 2, then the other can be $3, 4, 5, 6, 7, 8$, giving 6 possibilities.
     - The lesser of the pair is 3, then the other can be $4, 5, 6, 7$, giving 4 possibilities.
     - The lesser of the pair is 4, then the other can be $5, 6$, giving 2 possibilities.

     This gives $8+6+4+2 = 20$ possibilities altogether to if two numbers are not visible.
   - If three numbers are not visible, the possibilities are: 1,2,3; 1,2,4; 1,2,5; 1,2,6; 1,2,7; 1,3,4; 1,3,5; 1,3,6; 1,4,5; 2,3,4; 2,3,5; giving 11 possibilities.
   - If four numbers are not selected there is only one possibility: 1,2,3,4.

   This gives a total of $1+20+11+1 = 43$ ways for the coins to land in such a way that the sum of the numbers visible is at least 45. Since there is a total of 1024 possibilities, the required probability is $\frac{43}{1024}$.

   Answer: The probability is $\frac{43}{1024}$.

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5. (a) Each corner of each square face of the original cube has an isosceles right triangle removed. The long side of each of these triangles has length s. Thus, the two perpendicular legs have length $\ell = \dfrac{s}{\sqrt{2}}$. Each edge of the original cube consists of a side of one of the octagonal faces and two of the perpendicular legs of the isosceles right triangles. Hence, s can be found from

$$s + 2\left(\frac{s}{\sqrt{2}}\right) = s + \sqrt{2}s = s\left(1+\sqrt{2}\right) = 2 \Rightarrow s = \frac{2}{1+\sqrt{2}} = 2\left(\sqrt{2}-1\right)$$

   Answer: See the derivation above.

   (b) First note that

$$\ell = \frac{2\left(\sqrt{2}-1\right)}{\sqrt{2}} = \sqrt{2}\left(\sqrt{2}-1\right) = 2-\sqrt{2}$$

   The corners cut from the original cube are tetrahedra. These are pyramids whose base is the isosceles right triangle from the discussion above and height is equal to the length of the perpendicular legs of the triangles. Hence, the volume of one of the tetrahedra is

$$\begin{aligned} V_{\text{tetrahedron}} &= \frac{1}{3}\left(\frac{1}{2}\ell^2\right) \times \ell = \frac{1}{3}\left[\frac{1}{2}\left(2-\sqrt{2}\right)^2\right]\left(2-\sqrt{2}\right) \\ &= \frac{1}{3}\left[\frac{1}{2}\left(6-4\sqrt{2}\right)\right]\left(2-\sqrt{2}\right) = \frac{1}{3}\left(3-2\sqrt{2}\right)\left(2-\sqrt{2}\right) \\ &= \frac{1}{3}\left(10-7\sqrt{2}\right) \end{aligned}$$

   The volume of the original cube is $V_{\text{cube}} = 2^3 = 8$. There are eight corners removed from the cube, so the volume of the truncated cube is

$$8 - 8\left[\frac{1}{3}\left(10-7\sqrt{2}\right)\right] = \frac{8}{3}\left(3-10+7\sqrt{2}\right) = \frac{56}{3}\left(\sqrt{2}-1\right)$$

   Answer: The volume of the truncated cube is $\frac{56}{3}\left(\sqrt{2}-1\right)$.