BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2009

Junior & Senior Preliminary & Final Problems & Solutions

Junior Preliminary

1. The value of $2009 + 200.9 + 20.09 + 2.009 + 0.2009$ is:

   (A) 2231.3008 (B) 2232.108 (C) 2232.199 (D) 2231.2036 (E) 2232.1999

   Solution

   If we denote the given sum by S, then, by the usual method for computing a finite geometric sum, we have

$$\begin{aligned} S &= \quad 2009 + \quad 200.9 + 20.09 + 2.009 + 0.2009 \\ (0.1)S &= \qquad\qquad\quad 200.9 + 20.09 + 2.009 + 0.2009 + 0.02009 \end{aligned}$$

   Subtracting gives

$$(0.9)S = 2009 - 0.02009 \Rightarrow S = \frac{2009 - 0.02009}{0.9} = \frac{2008.97991}{0.9} = 2232.1999$$

   Careful calculation by the usual method gives the same result. Also note that a 9 in the ten thousandths place of a single number in the sum, as in 0.2009, tells us that the result must have a 9 in that place.

   Answer is (E).

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2. Tickets can only be ordered in bundles of 6 or 10. The minimum number of ticket bundles that are required to purchase exactly 52 tickets is:

   (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

   Solution

   Let x denote the number of bundles of 6, and let y denote the number of bundles of 10. We seek the solution of $6x + 10y = 52$ with both x and y being nonnegative integers, for which the total number of bundles $x + y$ is minimum. Obviously, the units digit of $6x$ must be 2, so $x = 2$ or $x = 7$ (since larger values of x give totals more than 52, making y negative). If $x = 2$, then $y = 4$ and $x + y = 6$. If $x = 7$, then $y = 1$ and $x + y = 8$. So the minimum number of bundles is 6.

   Answer is (B).

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3. Of the choices below the number that is less than its reciprocal is:

   (A) -1 (B) 1 (C) -2 (D) 2 (E) $-\frac{1}{3}$

   Solution

   Clearly, the reciprocal of -2 is $-1/2$ which is larger than -2.

   Answer is (C).

4. If 80 is divided by the positive integer n, the remainder is 4. The remainder when 155 is divided by n is:

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution

We have $80 = qn + 4$ where q is a positive integer and $0 \le 4 < n$, so $qn = 76 = 2^2(19)$. It follows that n can be 19, 38, or 76. We see $155 = 8(19) + 3 = 4(38) + 3 = 2(76) + 3$, so the remainder is 3 for all choices of n.

Answer is (D).

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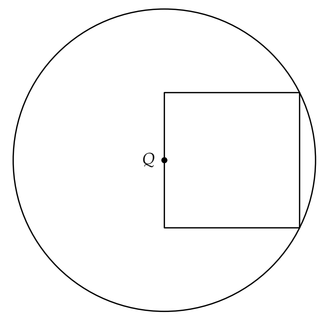
5. Point Q is the centre of a circle with a radius of 25 centimetres. A square is constructed with two vertices on the circle and the side joining the other two vertices containing the centre Q. The area of the square, in square centimetres, is:



(A) 125 (B) $125\sqrt{5}$ (C) 500

(D) 250 (E) $250\sqrt{5}$

Solution

If we let x denote the side length of the square, then, by Pythagoras' theorem,

$$\left(\tfrac{1}{2}x\right)^2 + x^2 = \tfrac{5}{4}x^2 = 25^2 \Rightarrow x^2 = 500$$

so the area of the square is $x^2 = 500$ square centimetres.

Answer is (C).

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6. If n is a positive integer, then $n! = n(n-1)(n-2)\cdots 2 \cdot 1$. For example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. The smallest positive integer that is not a divisor of 91! is:

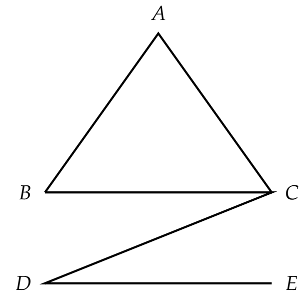
(A) 101 (B) 97 (C) 95 (D) 93 (E) 92

Solution

Observe that, $92 = 2 \times 46$, $93 = 3 \times 31$, $94 = 2 \times 47$, $95 = 5 \times 19$, $96 = 3 \times 32$, but 97 is prime and larger than 91, so the smallest non-factor among the numbers listed is 97.

Answer is (B).

7. In the diagram, triangle ABC is isosceles with $\angle ABC = \angle ACB$, and $\angle ACD$ is a right angle. Line segment DE is parallel to BC and $\angle CDE = 25°$. The measure of $\angle BAC$ is:



(A) 50° (B) 60° (C) 65°

(D) 90° (E) 115°

Solution

Observe that $\angle CDE = 25° = \angle BCD$. Then, $\angle ACB = 90° - 25° = 65° = \angle ABC$. The required measure of $\angle BAC$ is $180° - 2(65°) = 50°$.

Answer is (A).

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8. You play a game in which two six-sided dice are rolled. You win if the product of the two numbers rolled is odd or a multiple of 5. The probability that you win is:

(A) $\frac{1}{2}$ (B) $\frac{19}{36}$ (C) $\frac{5}{12}$ (D) $\frac{7}{12}$ (E) $\frac{25}{36}$

Solution

First note that the product is odd if and only if both numbers are odd, and there are 3 x 3 $= 9$ such outcomes. Next note that the product is a multiple of 5 exactly when at least one number is 5, and there are 12 such outcomes. Since there are six pairs with both properties, by the inclusion/exclusion principle there are $9 + 12 - 6 = 15$ outcomes in your favour. The 15 favourable outcomes, listed as ordered pairs of the numbers appearing on the two dice, are:

$$(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5),$$
$$(2,5), (4,5), (6,5), (5,2), (5,4), (5,6)$$

so the probability you win is $\frac{15}{36} = \frac{5}{12}$.

Alternative solution

There are 9 favourable outcomes that have an odd product, as above. The other favourable outcomes would be products that are even multiples of 5, which are:

$$(2,5), (4,5), (6,5), (5,2), (5,4), (5,6)$$

This gives a total of $9 + 6 = 15$ favourable outcomes.

Answer is (C).

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9. Tina runs 12 metres in the same time that Mark runs 7.5 metres. One day they ran around a 400-metre circular track. They leave the starting line at the same time and run in opposite directions. At the instant when Tina completes her second lap, the distance, in metres, that Mark is from the starting line would be:

(A) 50 (B) 100 (C) 125 (D) 150 (E) 200

Solution

In t units of time, Tina travels $12t$ metres and Mark travels $7.5t$ metres. After two laps Tina has traveled 800 metres, and this takes $\frac{800}{12} = \frac{200}{3}$ units of time. During this time Mark has traveled $7.5 \times \frac{200}{3} = 500$ metres, and so must be 100 metres from the starting line.

Answer is (B).

10. Given that a and b are digits from 1 to 9, the number of fractions of the form a/b, expressed in lowest terms, which are less than 1 is:

(A) 13 (B) 17 (C) 21 (D) 27 (E) 36

Solution

We could enumerate all such fractions systematically in the table below:

$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
	$\frac{2}{3}$		$\frac{2}{5}$		$\frac{2}{7}$		$\frac{2}{9}$
		$\frac{3}{4}$	$\frac{3}{5}$		$\frac{3}{7}$	$\frac{3}{8}$	
			$\frac{4}{5}$		$\frac{4}{7}$		$\frac{4}{9}$
				$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\frac{5}{9}$
					$\frac{6}{7}$		
						$\frac{7}{8}$	$\frac{7}{9}$
							$\frac{8}{9}$
1	2	2	4	2	6	4	6

The number of fractions in each column is given in the bottom row. The sum is 27 fractions.

Alternative solution

A more general approach is based on the famous totient function introduced by Euler. For each positive integer, n, we define $\phi(n)$ to be the number of positive integers less than or equal to n which are relatively prime to n (i.e., which have greatest common divisor 1 with n). This function has the following basic properties:

(1) If a and b are relatively prime, then $\phi(ab) = \phi(a)\phi(b)$; and

(2) If p is prime, then $\phi(p^n) = (p-1)p^{n-1}$.

In the present problem, we must calculate $\phi(1) + \phi(2) + \phi(3) + \cdots + \phi(9)$. Using the basic properties gives

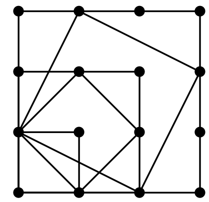
$$\phi(1) = 0, \phi(2) = (2-1) \cdot 2^0 = 1, \phi(3) = (3-1) \cdot 3^0 = 2, \phi\,(4) = \phi\left(2^2\right) = (2-1) \cdot 2^1 = 2$$
$$\phi(5) = (5-1) \cdot 5^0 = 4, \phi(6) = \phi(3)\phi(2) = 2, \phi(7) = (7-1) \cdot 7^0 = 6$$
$$\phi(8) = \phi\left(2^3\right) = (2-1) \cdot 2^2 = 4, \phi(9) = \phi\left(3^2\right) = (3-1) \cdot 3^1 = 6$$

So we calculate the sum to be

$$\phi(1) + \phi(2) + \phi(3) + \cdots + \phi(9) = 0 + 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 = 27$$

Answer is (D).

11. In the grid shown, the horizontal and vertical distance between adjacent dots is the same. The number of squares that can be formed for which all four vertices are dots in the grid is:

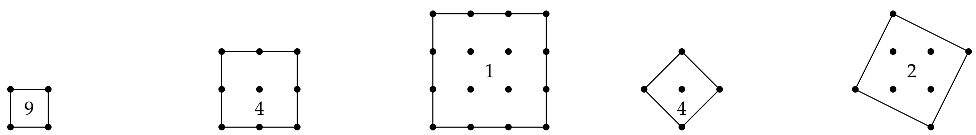


(A) 9 (B) 13 (C) 14

(D) 18 (E) 20

Solution

We identify the squares by orientation and by the number of dots encompassed. There are 9 horizontal squares with four dots, 4 horizontal squares with nine dots, and 1 horizontal square with sixteen dots. Next, there are 4 tilted squares with five dots, and 2 tilted squares with eight dots. This gives a total of $9 + 4 + 1 + 4 + 2 = 20$ such squares in the grid. See the diagrams below.



Answer is (E).

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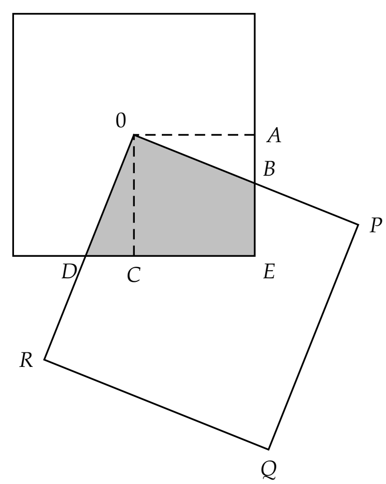
12. Two squares of side 1 are placed so that the centre of one square lies on a corner of the other, as shown in the diagram. The overlap of the two squares is shaded. The value of the shaded area is:

(A) $\frac{1}{4}$ (B) $\frac{1}{2\sqrt{2}}$ (C) $\frac{1}{2}$

(D) $\frac{1}{\sqrt{2}}$ (E) impossible to determine



Solution

Since square $OPQR$ is rotated about the centre of the other square, triangles OAB and OCD are congruent. As a result, the area of the shaded region is equal to the area of the square $OAEC$, which is

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Answer is (A).

Senior Preliminary

1. The expression

$$\frac{2010^2 + 2(2010)(2008) + 2008^2}{2010^2 - 2008^2}$$

equals:

(A) 4040100 (B) 4032064 (C) 45100 (D) 8407 (E) 2009

Solution

Factoring the numerator and denominator, and simplifying gives

$$\frac{2010^2 + 2(2010)(2008) + 2008^2}{2010^2 - 2008^2} = \frac{(2010 + 2008)^2}{(2010 + 2008)(2010 - 2008)} = 2009$$

Answer is (E).

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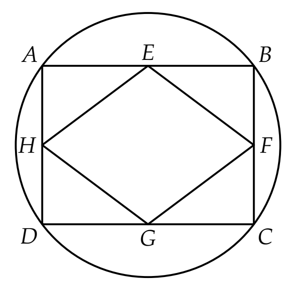
2. Quadrilateral $ABCD$ is inscribed in a circle. If $\overline{AB} = \overline{DC} = 8$ and $\overline{AD} = \overline{BC} = 6$, then the perimeter of the quadrilateral $EFGH$ formed by joining the midpoints E, F, G, and H of the sides AB, BC, CD, and DA is:



(A) 20 (B) 24 (C) 28

(D) 32 (E) 36

Solution

Since E and H are the midpoints of AB and AD, respectively, $\overline{AE} = 4$ and $\overline{AH} = 3$. Hence, by Pythagoras' theorem $\overline{HE} = 5$. By symmetry, $\overline{EF} = \overline{FG} = \overline{HG} = 5$. So the perimeter of quadrilateral $EFGH$ is 20.

Answer is (A).

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3. Jack walks up stairs one step at a time. Jill walks up stairs two steps at a time. Art, who likes to show-off, goes up three steps at a time. If each person starts with his or her left foot on the first step of the stairs, the first step on which all three will put their right foot is:

(A) 6 (B) 9 (C) 12 (D) 24 (E) Never happens

Solution

Jack's right foot treads on steps 2, 4, 6, 8, ... Jill's right foot treads on steps 3, 7, 11, 15,. . . . Art's right foot treads on steps 4, 10, 16, 22, ... Obviously, there is no step on which all three will tread with his or her right foot.

Answer is (E).

4. The value of the sum

$$1+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{36}+\frac{1}{72}+\frac{1}{216}+\frac{1}{432}+\cdots$$

is:

(A) $\frac{3}{2}$ (B) $\frac{9}{5}$ (C) 2 (D) $\frac{9}{4}$ (E) 3

Solution

Adding the terms in pairs, the sum can be written

$$S=\left(1+\frac{1}{2}\right)+\left(\frac{1}{6}+\frac{1}{12}\right)+\left(\frac{1}{36}+\frac{1}{72}\right)+\left(\frac{1}{216}+\frac{1}{432}\right)+\cdots=\frac{3}{2}\left(1+\frac{1}{6}+\frac{1}{36}+\frac{1}{216}+\cdots\right)$$

The expression in the parentheses form a geometric sum with a first term of 1 and ratio of successive terms equal to $\frac{1}{6}$. Hence, the sum is

$$S=\frac{3}{2}\left(\frac{1}{1-\frac{1}{6}}\right)=\frac{3}{2}\left(\frac{6}{5}\right)=\frac{9}{5}$$

Answer is (B).

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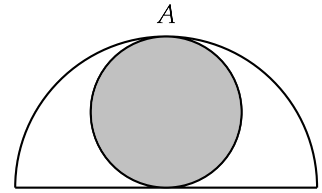
5. The shaded circle is tangent to the semicircle at the point A and the diameter of the semicircle at point B, the midpoint of the diameter of the semicircle. The ratio of the area of the shaded circle to the total area of the semicircle is:



(A) 1 (B) $\frac{2}{3}$ (C) $\frac{1}{2}$

(D) $\frac{1}{3}$ (E) $\frac{1}{4}$

Solution

If r is the radius of the semicircle, then the area of the semicircle is $\frac{1}{2}\pi r^2$, and the area of the shaded circle is $\pi\left(\frac{1}{2}r\right)^2$, so the ratio of the area of the shaded circle to the area of the semicircle is

$$\frac{\pi\left(\frac{1}{2}r\right)^2}{\frac{1}{2}\pi r^2}=\frac{1}{2}$$

Answer is (C).

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6. The area of the region enclosed by the graph of the equation $|x|+|y|=4$ for $-4\le x\le 4$ is:

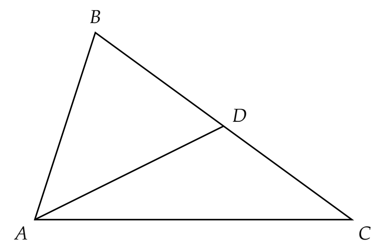
(A) 16 (B) 24 (C) 32 (D) 48 (E) 64

Solution

The graph of the equation is the square in the xy-plane with vertices at $(4,0)$, $(0,4)$, $(-4,0)$, and $(0,-4)$. The length of each side of the square is $\sqrt{4^2+4^2}=4\sqrt{2}$. So the area of the square is $\left(4\sqrt{2}\right)^2=32$.

Answer is (C).

7. In isosceles triangle ABC, sides AC and BC are equal. Point D lies on side BC such that both of the smaller triangles ABD and ACD are also isosceles. If AD and AB are equal, the measure of $\angle ABC$, in degrees, is:



(A) 70 (B) 64 (C) 60

(D) 80 (E) 72

Solution

Since triangle ABC is isosceles with AC and BC equal, $\angle ACB = 180 - 2\angle ABC$. Since triangle ABD is isosceles with AD and AB equal, $\angle DAB = 180 - 2\angle ABC$. Since triangle ACD is isosceles with AD and CD equal (obviously AD is shorter than AC), $\angle DAC = \angle ACB = 180 - 2\angle ABC$. Hence,

$$\angle BAC = \angle DAC + \angle DAB = 360 - 4\angle ABC = \angle ABC \Rightarrow 5\angle ABC = 360 \Rightarrow \angle ABC = 72$$

Answer is (E).

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8. If $p^2 + \dfrac{1}{p^2} = 7$ and $p > 0$, then the value of $p + \dfrac{1}{p}$ is:

(A) 3 (B) $\frac{1}{2}\left(3 - \sqrt{5}\right)$ (C) 7 (D) 9 (E) $\sqrt{\frac{1}{2}\left(7 + 3\sqrt{9}\right)}$

Solution

Squaring $p + \dfrac{1}{p}$ gives

$$\left(p + \frac{1}{p}\right)^2 = p^2 + 2 + \frac{1}{p^2} = \left(p^2 + \frac{1}{p^2}\right) + 2 = 9$$

Taking the square root gives

$$p + \frac{1}{p} = \pm 3$$

Since $p > 0$, take the positive root to give $p + \dfrac{1}{p} = 3$.

Answer is (A).

9. There are 6 white socks and 10 red socks jumbled up in a box. If 2 socks are taken out at random, the probability of having a matched pair is:

(A) $\frac{1}{2}$ (B) $\frac{5}{9}$ (C) $\frac{3}{5}$ (D) $\frac{2}{3}$ (E) None of these

Solution

The number of ways of choosing 2 socks from the 16 socks available is ${}_{16}C_2 = 120$. The number of ways of choosing 2 white socks is ${}_6C_2 = 15$ and the number of ways of choosing 2 red socks is ${}_{10}C_2 = 45$. Hence, the number of ways of choosing a matched pair is 60, and the probability is

$$\frac{\text{number of ways of choosing a matched pair}}{\text{number of ways of choosing 2 socks}} = \frac{15+45}{120} = \frac{1}{2}$$

Alternative solution

In choosing the socks one at a time, we require a white followed by a white or a red followed by a red. The required probability is

$$\left(\frac{6}{16}\right)\left(\frac{5}{15}\right) + \left(\frac{10}{16}\right)\left(\frac{9}{15}\right) = \frac{120}{240} = \frac{1}{2}$$

Answer is (A).

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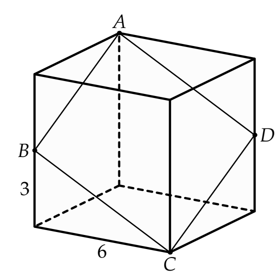
10. A cube with an edge length of 6 is cut by a plane to form a quadrilateral $ABCD$, where B and D are the midpoints of two edges of the cube. The area of the quadrilateral $ABCD$ is:



(A) 36 (B) $12\sqrt{6}$ (C) 45

(D) $18\sqrt{6}$ (E) 72

Solution

The quadrilateral $ABCD$ is a rhombus, not a square, with side length

$$\overline{AB} = \overline{BC} = \overline{CD} = \overline{DA} = \sqrt{3^2+6^2} = 3\sqrt{5}$$

The diagonals of the rhombus are $\overline{BD} = 6\sqrt{2}$ and $\overline{AC} = 6\sqrt{3}$. Since the diagonals of a rhombus are perpendicular, they divide the rhombus into four congruent right triangles. So the total area of the quadrilateral $ABCD$ is

$$4\left[\frac{1}{2}\left(3\sqrt{2}\right)\left(3\sqrt{3}\right)\right] = 18\sqrt{6}$$

Answer is (D).

11. If $x = \sqrt{3\sqrt{2\sqrt{3\sqrt{2\cdots}}}}$, then the value of x^3 is

(A) 6 (B) 8 (C) 9 (D) 18 (E) Answer is infinite

Solution

Square the expression for x to give

$$x^2 = 3\sqrt{2\sqrt{3\sqrt{2\cdots}}} = 3\sqrt{2x}$$

Squaring again gives

$$x^4 = 9\,(2x) = 18x$$

Hence,

$$x^4 - 18x = x\left(x^3 - 18\right) = 0$$

Since x obviously is not equal to zero, this gives $x^3 = 18$.

Answer is (D).

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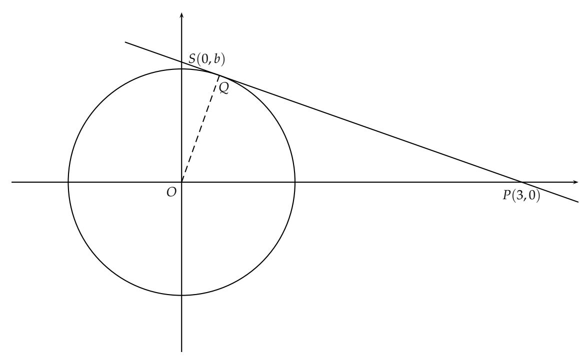
12. A line contains the point $(3, 0)$ and is tangent in the first quadrant to the unit circle centred at the origin. The y-intercept of this line is:

(A) 1 (B) $\frac{3}{\sqrt{8}}$ (C) $\frac{9}{8}$ (D) $\frac{5}{4}$ (E) $\frac{\sqrt{8}}{3}$

Solution



Define the points $P(3, 0)$, the point of tangency Q, the centre of the circle $O(0, 0)$, and the $S(0, b)$ where the line intersects the y-axis. (See the diagram.) Since the line PS is tangent to the circle, triangle OPQ is a triangle with a right angle at Q. Hence, by Pythagoras' theorem

$$\overline{PQ}^2 = 3^2 - 1^2 = 8 \Rightarrow \overline{PQ} = \sqrt{8}$$

Triangles OPQ and SPO are similar, so that

$$\frac{\overline{OS}}{\overline{OP}} = \frac{\overline{OQ}}{\overline{PS}} \Rightarrow \frac{b}{3} = \frac{1}{\sqrt{8}} \Rightarrow b = \frac{3}{\sqrt{8}}$$

Answer is (B).

Junior Final, Part A

1. In the game of Wombat the only scores are "gribbles" and "binks". Each "gribble" earns 4 points and each "bink" earns 5. The highest score that **cannot** be obtained is:

   (A) 11 (B) 13 (C) 17 (D) 21 (E) 23

   Solution

   Let g be the number of gribbles and b be the number of binks. Your score is $4g + 5b$ where g and b are non-negative integers. If we list the first few possible scores in a table we get

b \ g	0	1	2	3	4
0	0	4	8	12	16
1	5	9	13	17	21
2	10	14	18	22	26
3	15	19	23	27	31
4	20	24	28	32	36

   Note that 11 does not appear in the table, but the next four values, 12, 13, 14, and 15, do. By adding 1 to the value of g (one more gribble with a score of 4) the values 16, 17, 18, and 19 are obtained. In general, any value greater than 11 can be obtained by adding an appropriate integer to the value of g for one of the four values 12, 13, 14, or 15. Thus, 11 is the largest score that cannot be obtained.

   Comment: In general, if m and n have no common prime factors, then the largest number that cannot be obtained as a sum of multiples of m and n is $mn - (m + n)$. Using $m = 4$ and $n = 5$ we get $4 \times 5 - (4 + 5) = 11$.

   Answer is (A).

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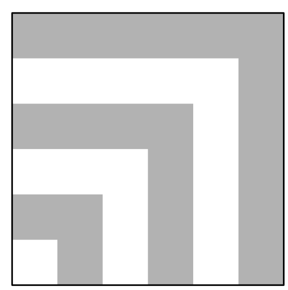
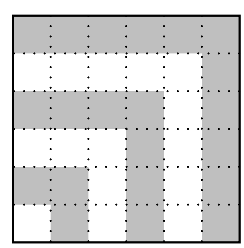
2. The gray and white strips in the target shown in the figure have equal width. A dart is thrown at the target where it sticks at a random location. The probability that the dart sticks in a gray strip is:



   (A) $\frac{3}{5}$ (B) $\frac{1}{2}$ (C) $\frac{19}{36}$

   (D) $\frac{5}{9}$ (E) $\frac{7}{12}$

   Solution



   Add lines to divide the target into squares, as shown. If we count squares, there are 21 gray squares and 36 squares in total, so the probability of sticking in a gray strip is:

$$\frac{21}{36} = \frac{7}{12}$$

   Answer is (E).

3. Jennifer has 21 coins consisting of dimes and quarters. If the dimes were quarters and the quarters were dimes, she would have \$1.05 less than she has now. Subtracting the number of dimes from the number of quarters gives:

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Solution

If d and q are the numbers of dimes and quarters, respectively, then $d + q = 21$. Further,

$$10d + 25q - (25d + 10q) = 105$$

Simplifying this gives $15\,(-d + q) = 105 \Rightarrow q - d = 7$. Note that it is not necessary to know that there are 21 coins.

Alternative solution

Each replacement of a quarter by a dime results in a loss of 15 cents. Hence, a loss of 105 cents means that there are $105/15 = 7$ more quarters than dimes in the initial collection.

Answer is (D).

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4. Antonino can run around a track in 5 minutes while Bill runs around the same track in 9 minutes. If Antonino and Bill start together, running in the same direction, the number of minutes it will take Antonino to gain one lap on Bill is:

(A) 10 (B) $10\frac{1}{4}$ (C) $10\frac{3}{4}$ (D) $11\frac{1}{4}$ (E) $11\frac{1}{2}$

Solution

If a and b are the total distance that Antonino and Bill, respectively, run in t minutes, and ℓ is the total distance around the track, then

$$a = \frac{\ell}{5}t \text{ and } b = \frac{\ell}{9}t$$

When Antonino has gained one lap on Bill

$$a = b + \ell \Rightarrow \frac{\ell}{5}t = \frac{\ell}{9}t + \ell \Rightarrow t = \frac{45}{4} = 11\tfrac{1}{4}$$

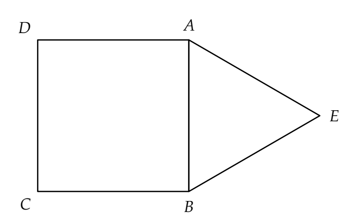
Alternative solution

Each minute Antonino gains $\frac{1}{5} - \frac{1}{9} = \frac{4}{45}$ lap on Bill. Hence, it will take $\frac{45}{4}$ minutes for Antonino to gain one lap on Bill.

Answer is (D).

---

5. In the figure, $ABCD$ is a square and ABE is an equilateral triangle. The measure of angle AED, in degrees, is:



(A) 10 (B) 15 (C) 18

(D) 20 (E) 30

Solution

Triangle ADE is isosceles, since $\overline{DA} = \overline{AE}$. Further, $\angle DAE = 90^\circ + 60^\circ = 150^\circ$. Hence

$$\angle AED = \tfrac{1}{2}\,(180^\circ - 150^\circ) = 15^\circ$$

Answer is (B).

6. The symbols Δ, Φ, Ψ, and ☺ represent integers. The sum of the values in each row and three of the columns is given. The value of $\Delta + \text{☺}$ is:

Δ	Φ	Φ	Φ	11
Δ	☺	☺	Φ	13
☺	☺	Δ	Δ	16
Φ	Ψ	Ψ	Ψ	14
15	12	14		

(A) 3 (B) 5 (C) 8

(D) 9 (E) 10

Solution

If we add the first three rows we get $4(\Delta + \Phi + \text{☺}) = 40 \Rightarrow \Delta + \Phi + \text{☺} = 10$. Adding the first two rows gives $2\Delta + 4\Phi + 2\text{☺} = 24 \Rightarrow \Delta + 2\Phi + \text{☺} = 12$. Subtracting gives $\Phi = 2$ so $\Delta + \text{☺} = 8$.

Alternative solution

Note that the sum of the third row is $2\text{☺} + 2\Delta = 16$. Therefore, $\Delta + \text{☺} = 8$.

Answer is (C).

---

7. The radius of the largest circle contained in a triangle with sides 3, 4, and 5 is:

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) $\frac{12}{5}$ (E) $\frac{4}{5}$

Solution

The circle so-defined is called the *incircle* of the given triangle. Drop perpendiculars of length r from the centre of the incircle to each side of the given triangle and connect the centre to each vertx of the triangle. Equating the areas of the large triangle with the sum of the areas of the three triangles just constructed, we find

$$\frac{1}{2}(5r) + \frac{1}{2}(3r) + \frac{1}{2}(4r) = \frac{1}{2}(4)(3) \Rightarrow 6r = 6 \Rightarrow r = 1$$

so the radius is 1.

Answer is (A).

---

8. If $2009 = a^b \times c$, where a, b, and c are all prime then the value of $\sqrt{\frac{a+b+c}{b}}$ is:

(A) 2 (B) 4 (C) 5 (D) 6 (E) 7

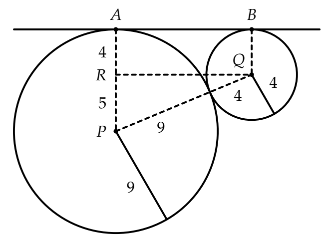
Solution

Factoring 2009 gives $2009 = (7^2)(41)$, so that $a = 7$, $b = 2$, and $c = 41$. Then $a + b + c = 50 = 2(5^2)$. Hence

$$\sqrt{\frac{a+b+c}{b}} = 5$$

Answer is (C).

9. Circles with centres at P and Q are tangent. The radius of the larger circle is 9 units and that of the smaller circle 4 units. The length of the common tangent $\overline{AB}$ is:



(A) 9 (B) 10 (C) 11

(D) 12 (E) 13

Solution

Triangle PRQ is a 5, 12, 13 right triangle. Hence, $\overline{AB} = \overline{RQ} = 12$.

Answer is (D).

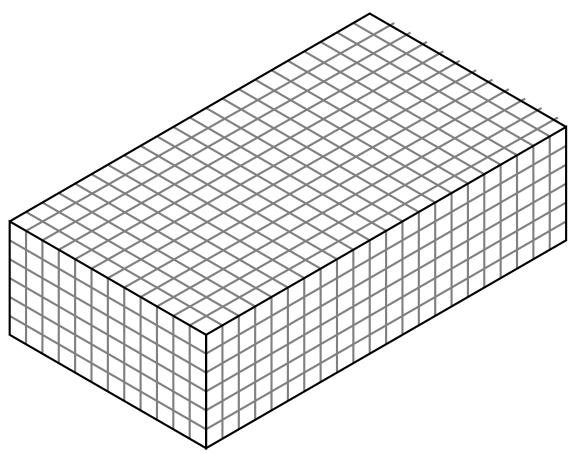
---

10. A $6\,\text{cm} \times 12\,\text{cm} \times 22\,\text{cm}$ rectangular block of wood is painted red and then cut into small cubes, each of which has a surface area of $6\,\text{cm}^2$. The number of small cubes that have red paint on exactly two faces is:

(A) 136 (B) 144 (C) 152 (D) 156 (E) 160

Solution

A cube with surface area 6 must be a unit cube. We cut the rectangular block into unit cubes, called "cubies", and count edge cubies (cubies on an edge but not at a vertex).

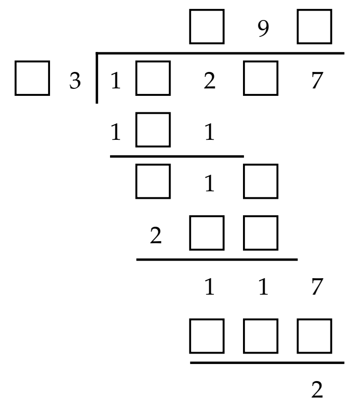


There are 4 edge cubies on each of the four sides of length 6, 10 on on each of the four sides of length 12, and 20 on each of the four sides of length 22 for a total of $4(4 + 10 + 20) = 136$.

Answer is (A).

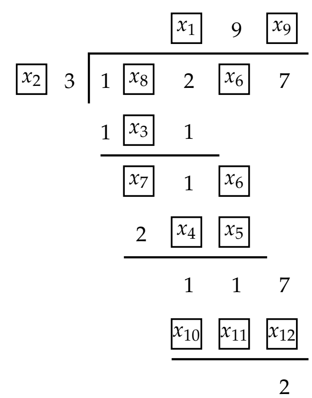
Junior Final, Part B

1. (a) Place any of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each of the boxes to correctly complete the long division below. Some digits may appear once, others may appear more than once, and some may not appear at all. (This problem may be done directly on the question sheet and full marks will be given for a correct answer. Explanation of your work is optional and will be graded only if there are any errors in the answer.)

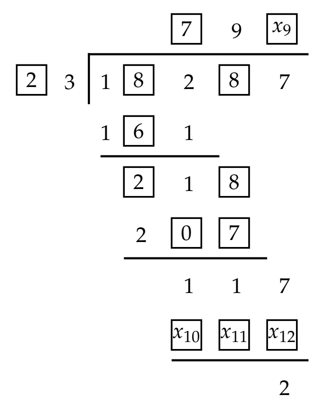


Solution

There is only one multiple of 3 which ends in 1, $3 \times 7 = 21$ so $x_1 = 7$. Only 3×23 lies between 100 and 200, as required, so $x_2 = 2$. This determines $x_3 = 6$.



Now $9 \times 23 = 207$ so x_4 and x_5 are determined: $x_4 = 0$ and $x_5 = 7$. But then $x_6 = 8$, $x_7 = 2$, and $x_8 = 8$.



...Problem 1 continued

Finally, we see that $x_{10} = x_{11} = 1$ and $x_{12} = 5$, so $x_9 = 5$.

```
                 [7]  9  [x9]
        [2] 3 ) 1 [8]  2  [8]  7
              1 [6]  1
              ---------
                [2]  1  [8]
                 2  [0] [7]
                 -----------
                     1   1    7
                  [x10][x11][x12]
                  ---------------
                              2
```

Alternative solution

Observe that the number beneath 117 must be 115, and also note that $115 = 23 \times 5$. Giving $x_{10} = 1$, $x_{11} = 1$, $x_{12} = 5$, $x_2 = 2$, and $x_9 = 5$. Next $9 \times 23 = 208$ giving $x_4 = 0$, $x_5 = 7$, $x_6 = 8$, and $x_7 = 2$. Finally, the only multiple of 3 that ends in 1 is $7 \times 3 = 21$. Hence, $x_1 = 7$, and, since $7 \times 23 = 161$, $x_3 = 6$ and $x_8 = 2$.

Answer:

```
                 [7]  9  [5]
        [2] 3 ) 1 [8]  2  [8]  7
              1 [6]  1
              ---------
                [2]  1  [8]
                 2  [0] [7]
                 -----------
                     1   1   7
                    [1] [1] [5]
                    -----------
                             2
```

(b) List the positive integers n for which $\dfrac{420}{3n+1}$ is an integer.

Solution

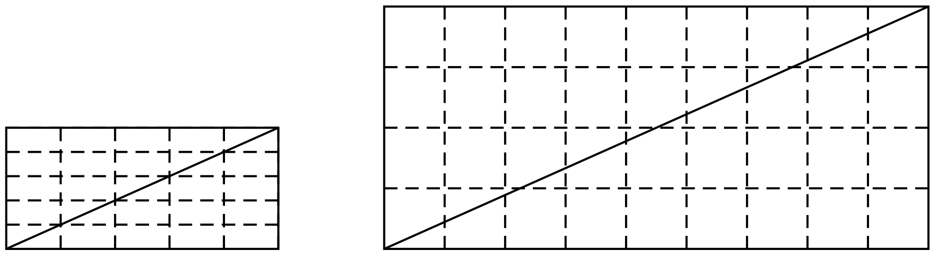
Since $420 = 2^2 \times 3 \times 5 \times 7$, and we want $3n+1$ to be a divisor of 420, it follows that $3n+1$ must be made from some combination of these divisors other than 3. There are twelve possible divisors using the factorization $2^2 \times 5 \times 7$. They are 1 ($n = 0$), 2, 4 ($n = 1$), 5, 7 ($n = 2$), 10 ($n = 3$), 14, 20, 28 ($n = 9$), 35, 70 ($n = 23$), and 140. The divisors that are equal to $3n+1$ for some integer n are indicated with the value of n. Since n must be positive, the possible values of n are $n = 1$, $n = 2$, $n = 3$, $n = 9$, and $n = 23$.

Answer: The possible values of the integer n are: 1, 2, 3, 9, and 23

2. The floor in a rectangular room in Hernando's house has 20 square tiles along its width and 45 square tiles along its length. Hernando draws a diagonal from one corner of the room to the opposite corner. How many tiles does the diagonal cross?

Solution

Since $\frac{20}{45} = \frac{5\cdot 4}{5\cdot 9}$, divide the room into $5^2 = 25$ rectangles, 5 of which lie on the main diagonal, consisting of 4×9 squares as shown below.



The diagonal of each rectangle passes through 9 squares horizontally and 4 squares vertically. Since the left bottom corner square is counted twice, the diagonal of each rectangle passes through $9 + 4 - 1 = 12$ squares, and therefore the diagonal of the room crosses $12 \times 5 = 60$ squares.

Answer: The diagonal crosses through 60 squares.

---

3. Children have been bringing cookies for a school cookie sale, and your class is keeping track of the offerings. They find that $\frac{3}{10}$ of the cookies contain (among other things) oatmeal, $\frac{1}{2}$ of them have (among other things) chocolate chips, and $\frac{3}{28}$ of them have both oatmeal and chocolate chips. If 172 of the cookies have neither oatmeal nor chocolate chips, determine the total number of cookies offered.

Solution

Let t be the total number of cookies, o the number of cookies with oatmeal, and c the number of cookies with chocolate chips. By the inclusion-exclusion principle (or see the Venn diagram below),

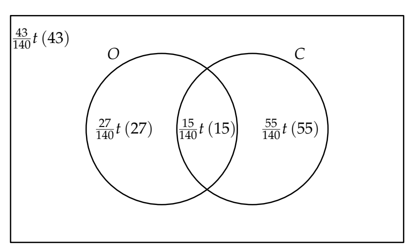
$$o + c = t\left(\frac{3}{10} + \frac{1}{2} - \frac{3}{28}\right) = \frac{97}{140}t$$

Hence, the total number of cookies is given by

$$t = 172 + \frac{97}{140}t \Rightarrow t = \frac{140}{43} \times 172 = 560$$

Alternative solution

The lowest common multiple of 10, 2, and 28 is 140. Suppose that the total number of cookies is $t = 140$. A Venn diagram summarizing the information given about the proportion of each type of cookie, where O is the set of all oatmeal cookies and C is the set of all chocolate chip cookies, is shown. This leaves 43 cookies having neither oatmeal or chocolate chips. We require that there be 172 such cookies, as in 4×43. Therefore, the total number of cookies is $4 \times 140 = 560$.



Answer: The total number of cookies is 560.

4. How many three digit whole numbers have digits that when multiplied together give a product that is greater than 60 and less than 65?

Solution

It is required to find triples of digits A, B, and C from 1 to 9 that satisfy the inequality

$$60 < A \times B \times C < 65$$

None of the digits can be 0, otherwise the product would be zero. Now, consider prime factorizations of the numbers strictly between 60 and 65.

61: is prime, hence no triples exist for 61.

62: $62 = 2 \times 31$ and 31 is too big to be a digit, hence, no triples exist for 62.

63: $63 = 3^2 \times 7$. Hence, two triples exist $\{3, 3, 7\}$ and $\{1, 7, 9\}$.

64: $64 = 2^6$. Hence, three triples exist $\{1, 8, 8\}$, $\{4, 4, 4\}$ and $\{2, 4, 8\}$.

There are 5 triples, where A,B, and C may be chosen in any order from the list: {1, 7, 9}, {1, 8, 8}, {2, 4, 8}, {3, 3, 7}, {4, 4, 4}. For the triples {1, 7, 9} and {2, 4, 8} with three distinct digits there are $3! = 6$ possible numbers. For the triples {1, 8, 8} and {3, 3, 7} with two distinct digits there are 3 possible numbers, one for each possible position of the non-repeated number. For the triple {4, 4, 4} there is only one number. This gives a total of $6 + 6 + 3 + 3 + 1 = 19$ numbers.

Alternative solution

Observe that $3 \times 3 \times 3 = 27$ and hence, at least one of the digits must be greater than or equal to 4. Consider cases based on the largest digit in the three digit whole number, as follows:

Case 1: The largest digit is 4.
The largest possible product is $4 \times 4 \times 4 = 64$. This gives one solution (444).

Case 2: The largest digit is 5.
This is impossible since 5 is not a factors of 61, 62, 63, or 64.

Case 3: The largest digit is 6.
Likewise, 6 is not a factors of 61, 62, 63, or 64.

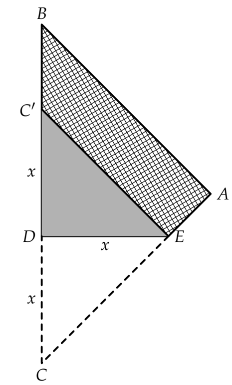
Case 4: The largest digit is 7.
Since $7 \times 9 = 63$, there are solutions. Since 7 is the largest digit, express 63 as $7 \times 3 \times 3$. This gives three solutions (733, 373, and 336).

Case 5: The largest digit is 8.
Since $8 \times 8 = 64$, there are solutions. Observe that $8 \times 8 \times 1 = 64$ giving three solutions (188, 818, and 881). Further, $8 \times 4 \times 2 = 64$ giving $6 = 3!$ additional solutions (248, 284, 428, 482, 824, and 842).

Case 6: The largest digit is 9.
Since $9 \times 7 = 9 \times 7 \times 1 = 63$ there are six more solutions (179, 197, 719, 791, 917, and 971).

Answer: There are 19 such numbers.

5. An isosceles right triangle ABC with legs of length 2 cm is cut from a sheet of paper that is cross-hatched on one side and is solid gray on the other. The triangle is folded by moving the vertex C to position C' on side BC. If the cross-hatched area and the solid gray area are equal, determine the distance between B and C'.



Solution

Let x be the length shown in the diagram, then the desired length is

$$\overline{BC'} = \overline{BC} - 2x$$

where $\overline{BC} = \sqrt{8} = 2\sqrt{2}$. Since the two isosceles right triangles CDE and $C'DE$ and the quadrilateral $ABC'E$ all have the same area, each has an area equal to one-third of the area of the original isosceles right triangle. So, the area of either of the isosceles right triangles CDE and $C'DE$ is

$$\frac{1}{2}x^2 = \frac{1}{3}\left(\frac{1}{2}\cdot 2\cdot 2\right) = \frac{2}{3} \Rightarrow x = \frac{2}{\sqrt{3}}$$

Hence, the required length is

$$\overline{BC'} = 2\sqrt{2} - 2\left(\frac{2}{\sqrt{3}}\right) = 2\sqrt{2} - \frac{4\sqrt{3}}{3} = \tfrac{2}{3}\left(3\sqrt{2} - 2\sqrt{3}\right) \text{ cm}$$

Answer: The distance between B and C' is $\frac{2}{3}\left(3\sqrt{2} - 2\sqrt{3}\right)$ cm

Senior Final, Part A

1. The symbols Δ, Φ, Ψ, and ☺ represent integers. The sum of the values in each row and three of the columns is given. The value of Δ is:

Δ	Φ	Φ	Φ	11
Δ	☺	☺	Φ	13
☺	☺	Δ	Δ	16
Φ	Ψ	Ψ	Ψ	14
15	12	14		

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5

Solution

Subtracting the second row from the third row gives $\Delta - \Phi = 3$. Multiplying this by 3 and adding to the first row gives $4\Delta = 20$. Therefore, $\Delta = 5$.

Answer is (E).

2. Antonino can run around a track in 5 minutes while Bill runs around the same track in 9 minutes. If Antonino and Bill start together, running in the same direction, the number of minutes it will take Antonino to gain one lap on Bill is:

(A) 10 (B) $10\frac{1}{4}$ (C) $10\frac{3}{4}$ (D) $11\frac{1}{4}$ (E) $11\frac{1}{2}$

Solution

If a and b are the total distance that Antonino and Bill, respectively, run in t minutes, and ℓ is the total distance around the track, then

$$a = \frac{\ell}{5}t \text{ and } b = \frac{\ell}{9}t$$

When Antonino has gained one lap on Bill

$$a = b + \ell \Rightarrow \frac{\ell}{5}t = \frac{\ell}{9}t + \ell \Rightarrow t = \frac{45}{4} = 11\tfrac{1}{4}$$

Alternative solution

Each minute Antonino gains $\frac{1}{5} - \frac{1}{9} = \frac{4}{45}$ lap on Bill. Hence, it will take $\frac{45}{4}$ minutes for Antonino to gain one lap on Bill.

Answer is (D).

---

3. Five straight lines are drawn on the plane. The maximum possible number of intersection points of the five lines is:

(A) 5 (B) 6 (C) 10 (D) 15 (E) 20

Solution

To maximize the number of intersections, each new line is added in such a way that it intersects with each of the previous lines. Then the maximum possible number of intersections is $0 + 1 + 2 + 3 + 4 = 10$.

Answer is (C).

4. The number 2009 can be expressed as the sum of n ($n \geq 2$) consecutive odd integers in several ways. The smallest possible value of n is:

(A) 5 (B) 7 (C) 21 (D) 41 (E) 49

Solution

Note that $2009 = 7^2 \times 41$. Each divisor of 2009 gives a sequence of odd integers which add to 2009:

$$\begin{aligned} 7 \times 287 &= 281 + 283 + 285 + 287 + 289 + 291 + 293 \\ 41 \times 49 &= 9 + \cdots + 47 + 49 + 51 + \cdots + 89 \\ 49 \times 41 &= -7 + \cdots + 39 + 41 + 43 + \cdots + 89 \\ 287 \times 7 &= -279 + \cdots + 5 + 7 + 9 + \cdots + 293 \\ 2009 \times 1 &= -2007 + \cdots - 1 + 1 + 3 + \cdots + 2009 \end{aligned}$$

The smallest number of terms is 7.

Alternative solution I

Observe that n must be odd since the sum of the n odd integers is 2009, which is odd. The middle integer in the ordered sequence of n consecutive odd integers will be the average of the integers and hence

$$2009 = n \times \text{the middle integer}$$

Therefore, the smallest possible value of n is the smallest possible divisor (other than 1) of 2009. Note that 2009 is not divisible by 3 or 5, and is divisible by 7. So the smallest possible value of n is 7.

Alternative solution II

Let $2k + 1$ be the first odd number in the sequence of n consecutive odd integers. Then the sum of the integers is

$$\underbrace{(2k+1) + (2k+3) + \cdots + (2k+2n-1)}_{n \text{ terms}} = 2kn + n^2 = 2009$$

Solving for k gives

$$k = \frac{2009 - n^2}{2n}$$

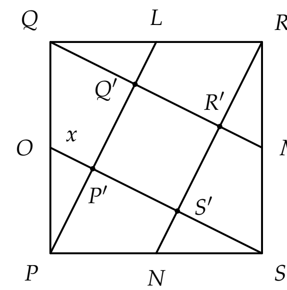
For k to be an integer, $2009 - n^2$ must be an even multiple of n and n must be a factor of 2009. Since $2009 = 7^2 \times 41$, the only possibilities for n are $n = 1$, $n = 7$, $n = 41$, $n = 49$, $n = 287$, and $n = 2009$. Since $n \geq 2$, the smallest possible value of n is $n = 7$. In this case,

$$k = \frac{2009 - 49}{14} = \frac{1960}{14} = 140 \Rightarrow 2k + 1 = 281$$

giving the first sequence shown above.

Answer is (B).

5. In the square $PQRS$ shown in the figure, the points L, M, N, and O are the midpoints of the sides. A smaller square $P'Q'R'S'$ is formed inside the larger square. The ratio of the area of square $P'Q'R'S'$ to the area of square $PQRS$ is:



(A) 1 : 5 (B) 1 : 4 (C) 1 : 3

(D) 1 : 2 (E) 2 : 5

Solution

Let $2a$ be the side length of the large square. Observe that the angles at the internal intersection points P', Q', R', and S' are right angles. This means that the triangles $PP'O$, $PQ'Q$, and PQL are similar, so that if $\overline{OP'} = x$, then $\overline{QQ'} = 2x$ and $\overline{PQ'} = 4x$. Hence, $\overline{PL} = 5x$ and by Pythagoras' theorem

$$5x = \sqrt{a^2 + 4a^2} = \sqrt{5}a \Rightarrow x = \frac{1}{5}\sqrt{5}a$$

Since $\overline{Q'P'} = \overline{PQ'} - \overline{PP'} = 4x - 2x = 2x$, the area of the square $P'Q'R'S'$ is

$$4x^2 = \frac{4}{5}a^2$$

The area of the square $PQRS$ is $4a^2$ so the required ratio of the area of the squares is 1 : 5.

Note: Adding triangle LQQ' to trapezoid $Q'QOP'$ gives a square congruent to square $P'Q'R'S'$. The same is true for each of the other three trapezoid-triangle pairs. Hence,

$$\text{Area } PQRS = 5\text{Area } P'Q'R'S'$$

Answer is (A).

---

6. The equation $x^2 + Bx + 2 = 0$ has only one root. The product of the possible values of B is:

(A) 8 (B) -8 (C) $2\sqrt{2}$ (D) $-2\sqrt{2}$ (E) -4

Solution

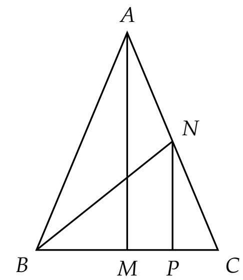
The roots of $x^2 + Bx + 2 = 0$ are

$$x = \frac{-B \pm \sqrt{B^2 - 4 \times 2}}{2}$$

If there is only one root then $B^2 - 8 = 0 \Rightarrow B = \pm 2\sqrt{2}$. The product of these two roots is -8.

Answer is (B).

7. In ΔABC we have $\overline{AB} = \overline{AC}$, $\overline{AN} = \overline{NC}$, and $\overline{BM} = \overline{MC}$ with $\overline{MC} = 5$ and $\overline{AM} = 12$. The shortest distance from point N to line segment BC is:



(A) 6 (B) $\frac{13}{2}$ (C) $\frac{169}{24}$

(D) 8 (E) 10

Solution

The shortest line segment from N to BC is the perpendicular from N to BC. Let P be the point where the perpendicular from N intersects BC. Then $\overline{NP}$ is the distance desired. Since $\overline{AB} = \overline{AC}$, triangle ABC is isosceles and, since $\overline{BM} = \overline{MC}$, M is the midpoint of BC. Therefore, AM is perpendicular to BC, and with $\overline{AM} = 12$ and $\overline{MC} = 5$ Pythagoras' theorem gives $\overline{AC} = 13$. Hence, $\overline{NC} = \frac{13}{2}$. Finally, triangles ACM and NCP are similar so that

$$\frac{\overline{NP}}{\overline{NC}} = \frac{\overline{AM}}{\overline{AC}} \Rightarrow \frac{\overline{NP}}{\frac{13}{2}} = \frac{12}{13} \Rightarrow \overline{NP} = 6$$

Answer is (A).

---

8. X and Y are positive integers. The sum of the digits of X is 53, and the sum of the digits of Y is 47. If the addition of X and Y involves exactly 5 carries, the sum of the digits of $X + Y$ is:

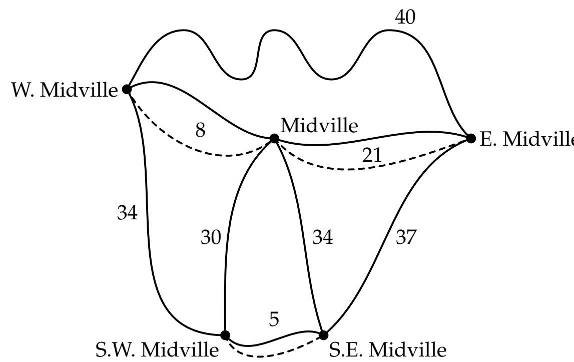
(A) 45 (B) 55 (C) 95 (D) 100 (E) Impossible to determine.

Solution

For each carry in the calculation of $X + Y$ the sum of digits is reduced from 10 to 1, that is by 9. Consequently, the sum of the digits of $X + Y$ is $53 + 47 - 9 \times 5 = 55$.

Answer is (B).

9. The Middle Okanagan Regional District has just resurfaced a local road system, and now the yellow strip down the middle of the roads must be repainted. The District Manager would like to have the truck used for this purpose to travel the shortest distance possible. A road map of the local road system is shown, with distances given in kilometres. The truck is garaged in Midville and must return there when the job is done. The number of kilometres the truck must travel in order to cover each road in the system at least once, and return to its starting point in Midville, could be:



(A) 209 (B) 214 (C) 230 (D) 243 (E) 254

Solution

We have a map in which all the towns except Midville have three roads entering them (Midville has four). In order to traverse all the roads once and start and end in the same town it is necessary that all the towns have an even number of roads entering them. To solve the problem we can introduce "dummy" roads of the same length and between the same town (representing where the truck travels but does not paint) so that all the towns have an even number of roads entering them and the total length of the added roads is a minimum. This is clearly done by duplicating the three shortest roads, as shown by the extra dashed roads in the diagram, making the length of the trip

$$40 + 2 \times 8 + 2 \times 21 + 34 + 30 + 34 + 37 + 2 \times 5 = 243$$

Answer is (D).

---

10. A point P is chosen in the first quadrant so that the lines from P to the point $(1, 0)$ and from P to the point $(-1, 0)$ are perpendicular. The shortest distance from any such point P to the point $(1, 1)$ is:

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\sqrt{2} - 1$ (D) $\frac{1}{5}\sqrt{5}$ (E) $2 - \sqrt{2}$

Solution

The locus of points making a right angle from two fixed points is a circle with the two fixed points on a diameter. So here the circle is centred at the origin and has radius 1. The shortest distance from the circle to $(1, 1)$ is the distance from the centre of the circle, the origin, to $(1, 1)$ minus the radius. So the shortest distance is $\sqrt{2} - 1$.

Answer is (C).

Senior Final, Part B

1. Of the students in a class 17 can ride a bicycle, 13 can swim, and 8 can ski. No student is able to perform all three of these activities. All the bicyclists, swimmers, and skiers have received a grade of C or better in the class. Six students in the class received a grade less than C. Determine the smallest possible number of students in the class.

Solution

By the inclusion-exclusion principle, the total number of students in the class will be as small as possible if everyone who does at least one of the activities does two. Let A be the set of students who cycle and swim, B be the set of students who cycle and ski, and C be the set of students who swim and ski. Then, by the inclusion-exclusion principle

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Since 17 students cycle, $|A \cup B| = 17$ and, since no students cycles, swims, and skis, $|A \cap B| = 0$. Hence

$$|A \cup B| = |A| + |B| = 17$$

In the same way, $|A| + |C| = 13$ and $|B| + |C| = 8$. Solving for $|A|$, $|B|$, and $|C|$ gives $|A| = 11$, $|B| = 6$, and $|C| = 2$. So, the total number of athletes is 19. Since there are at least 6 non-athletes, the smallest possible number of students in the class is 25.

Alternative solution

Let X be the number of students who do exactly one of the activities and Y the number of students who do exactly two of the activities. We know that no students do all three of the activities. Then the total number of students in the class is at least $X + Y + 6$. Further, adding the numbers of bicyclists, swimmers, and skiers gives

$$X + 2Y = 17 + 13 + 8 = 38$$

since any student who does two of the activities is counted twice. So it is required to make $X + Y + 6$ as small as possible subject to the constraint that $X + 2Y = 38$. Solving for Y in the constraint and substituting into the expression for the total number of students gives

$$X + \left(19 - \tfrac{1}{2}X\right) + 6 = \tfrac{1}{2}X + 25$$

This is smallest when $X = 0$ (every student does exactly two of the activities) since any nonnegative value of X makes it larger. Hence, the smallest possible value is 25.

Answer: The smallest possible number of students in the class is 25.

2. (a) Use the fact that the last two digits of 3^9 are 83 to determine the last two digits of 3^{10}.

Solution

If 3^9 has last two digits 83, then $3^{10} = 3^9 \times 3 = \ldots 83 \times 3 = \ldots 49$ has 49 as the last two digits.

Answer: The last two digits of 3^{10} are 49.

(b) Determine the last two digits of 3^{20}.

Solution

Since $3^{20} = (3^{10})^2 = (\ldots 49)^2 = \ldots 01$, then 3^{20} has 01 as the last two digits.

Answer: The last two digits of 3^{20} are 01.

(c) Determine the last two digits of 3^{2009}.

Solution

Rewriting 3^{2009} using previous powers of three, we get:

$$3^{2009} = (3^{20})^{100} \times 3^9 = (\ldots 01)^{100} \times (\ldots 83) = (\ldots 01) \times (\ldots 83) = \ldots 83$$

Hence, the last two digits of 3^{2009} are 83.

Answer: The last two digits of 3^{2009} are 83.

3. Consider the equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$$

where x, y, and z are positive integers.

(a) Find all of the solutions to the equation above for which $x \le y \le z$.

Solution

If $x = 1$ in the equation with $x \le y \le z$, then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} > 1$. Hence, $x \ge 2$.

When $x = 2$ the equation becomes $\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$, with $2 \le y \le z$. If $y = 2$ with $y \le z$, then $\frac{1}{y}+\frac{1}{z} > \frac{1}{2}$. Hence, $y \ge 3$. Now

$$y = 3 \Rightarrow \frac{1}{z}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6} \Rightarrow z = 6 \text{ and } y = 4 \Rightarrow \frac{1}{z}=\frac{1}{2}-\frac{1}{4}=\frac{1}{4} \Rightarrow z = 4$$

If $y > 4$ with $y \le z$, then $\frac{1}{y}+\frac{1}{z} < \frac{1}{2}$. So the two cases above exhaust all of the possibilities for $x = 2$.

When $x = 3$ the equation becomes $\frac{1}{y}+\frac{1}{z}=\frac{2}{3}$, with $3 \le y \le z$. Now

$$y = 3 \Rightarrow \frac{1}{z}=\frac{2}{3}-\frac{1}{3}=\frac{1}{3} \Rightarrow z = 3$$

If $y \ge 4$ with $y \le z$, then $\frac{1}{y}+\frac{1}{z} \le \frac{1}{2} < \frac{2}{3}$. So the single case above is the only possibility for $x = 3$.

If $x \ge 3$ with $x \le y \le z$, then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} < 1$. Hence, all possibilities have been exhausted and the solutions to the equation above are: $(x, y, z) = (2, 3, 6)\,, (2, 4, 4)\,, (3, 3, 3)$.

Answer: The solutions are: $(x, y, z) = (2, 3, 6), (2, 4, 4), (3, 3, 3)$

(b) Determine the total number of solutions to the equation above with the restrictions from part (a) removed.

Solution

Removing the restriction that $x \le y \le z$, there are $3! = 6$ choices for the ordering of the values in the triple $(2, 3, 6)$, 3 choices for the triple $(2, 4, 4)$, and only 1 for $(3, 3, 3)$. This gives a total of 10 solutions to the equation.

Answer: There are 10 possible solutions.

4. Three 1's, three 0's, and three -1's are placed in the 3×3 grid shown in such a way that the sign, or value if it is zero, of the sum in each row, each column, and one diagonal is as shown. The value of the sum of the numbers along the other diagonal must be an integer between -3 and 3, inclusive.

			+
			−
			0
0	+	−	0

(a) Draw one grid, with numbers filled in according to the conditions given, for which the sum along the other diagonal is 3.

Solution

The grid shown has a sum of 3 along the other diagonal:

0	0	1	+
-1	1	-1	−
1	0	-1	0
0	+	−	0

Answer: A grid giving a sum of 3 along the other diagonal is shown.

(b) Draw one grid, with numbers filled in according to the conditions given, for which the sum along the other diagonal is 2.

Solution

The grid shown has a sum of 2 along the other diagonal:

0	1	0	+
-1	1	-1	−
1	0	-1	0
0	+	−	0

Answer: A grid giving a sum of 2 along the other diagonal is shown.

(c) Determine which of the values 1, 0, and -1 can be obtained as the sum along the other diagonal. For any value that is possible, draw one grid, with numbers filled in according to the conditions given, for which the sum along the other diagonal is that value. For any value that is impossible, demonstrate that the value is impossible.

Solution

The grids below have sums of 1, 0, and -1, respectively, along the other diagonal

1	0	1	+
-1	0	-1	−
0	1	-1	0
0	+	−	0

1	1	0	+
-1	0	-1	−
0	1	-1	0
0	+	−	0

1	1	-1	+
-1	0	0	−
0	1	-1	0
0	+	−	0

Answer: Grids giving sums of 1, 0, and -1 along the other diagonal are shown.

5. Find the area of the region inside the circle $x^2 + y^2 = 6x$ but outside the circle $x^2 + y^2 = 27$.

Solution

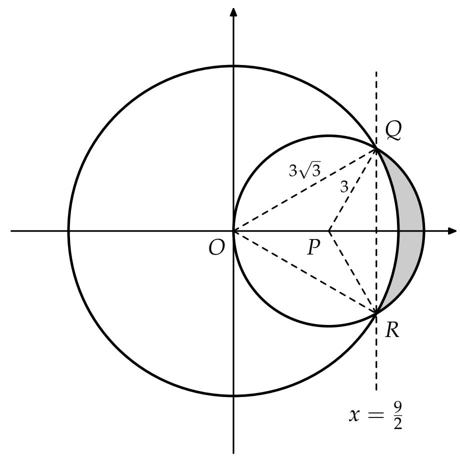
Rewrite the first equation as

$$(x-3)^2 + y^2 = 9$$

The graph is a circle with radius 3 centred at $(3, 0)$. The second is the equation of a circle with radius $3\sqrt{3}$ centred at the origin. These circles intersect where

$$6x = 27 \Rightarrow x = \frac{9}{2}$$

The required area is the area of the crescent outside of the smaller circle and inside the larger circle. See the diagram.



This area is the difference between the areas of the segments of the two circles to the right of the line $x = \frac{9}{2}$. The area of the segment of the smaller circle to the right of $x = \frac{9}{2}$ is the difference between the area of the circular sector PQR and the triangle PQR, which is:

$$A_1 = \frac{1}{2}(9)\left(\frac{2}{3}\pi\right) - \frac{1}{2}\left(\sqrt{27}\right)\left(\frac{3}{2}\right) = 3\pi - \frac{3}{4}\sqrt{27}$$

The area of the segment of the larger circle to the right of $x = \frac{9}{2}$ is the difference between the area of the circular sector OQR and the triangle OQR, which is

$$A_2 = \frac{1}{2}(27)\left(\frac{1}{3}\pi\right) - \frac{1}{2}\left(\sqrt{27}\right)\left(\frac{9}{2}\right) = \frac{9}{2}\pi - \frac{9}{4}\sqrt{27}$$

The difference between the areas is

$$A_1 - A_2 = \left(3\pi - \frac{3}{4}\sqrt{27}\right) - \left(\frac{9}{2}\pi - \frac{9}{4}\sqrt{27}\right) = \frac{3}{2}\left(\sqrt{27} - \pi\right)$$

Answer: The area is $\frac{3}{2}\left(\sqrt{27} - \pi\right)$